CHAPTER 6
REPRESENTATION

6.1. In order to achieve an axiomatization of the informational approach in the easiest way, I begin by reasoning informally under some schematizations.

Though in our everyday practice a reliability degree (from 0 to 1, say) can be associated to any piece of information belonging to our knowledge, for the moment we only consider the extreme values, thus sharply opposing *known* to *unknown*.

For instance let me consider three consecutive tosses of a well-balanced coin. Since any outcome is either head (A) or tail (B); the possibility space Ω of such a sequence contains eight alternatives (either *AAA* or *AAB* or *ABA* or *ABB* or *BAA* or *BAB* or *BBA* or *BBB*).

I call "basic statute" the piece of information $k°$ defining Ω and "acquirement" any subsequent piece of information k' concerning the same Ω. Simple but general evidence is that any acquirement corresponds to the preclusion of some alternative (and vice versa). For instance to learn that two consecutive but not better specified tosses gave the same but not better specified outcome precludes *ABA* and *BAB*. I call "free" the non-precluded alternatives and "statute (of g at t about Ω)" the conjunction $k=k°\&k'$, that is the whole information about Ω possessed by the knower g at the moment t.

6.2. In order not to waste time with analyses of negligible interest, the coherence of the statute is presupposed; this only means that the eventual incoherence of a statute must be explicitly remarked.

The existence of a statute does not entail the ontological reality of the universe it describes. Since both information and imagination are mental things, we can speak of Plato (whose historical existence is sure) exactly as we can speak of Homer (whose historical existence is controversial) or of Polyphemus. Analogously, in order to reason on our Ω, it is not at all necessary to perform materially the tosses.

Of course a radical difference distinguishes the actual world from fictitious ones, giving the former a quite privileged role; yet this topic will be analysed in Chapter 13.

6.3. The informational approach is strongly helped by the intensional representation ® whose main lines I am about to illustrate.

6.3.1. Figure 6.1

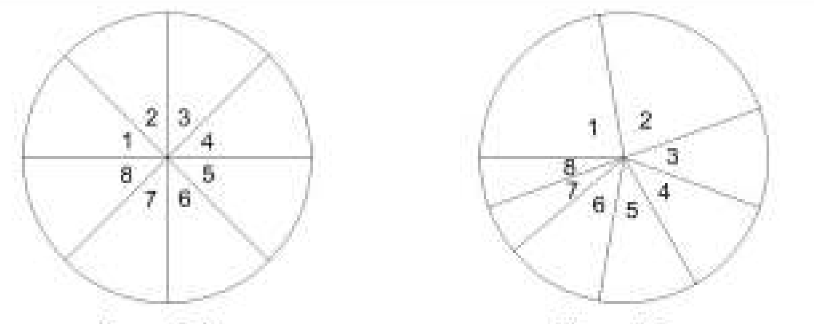

Figure 6.1 Figure 6.2

illustrates the first basic idea of ®: representing $k°$ by a circle partitioned in as many sectors as the alternatives of the respective possibility space Ω: so, with reference to the example above, the sector 1 represents *AAA* et cetera, till the sector 8 for *BBB*.

For the sake of concision "sector representing the outcome so and so" can be abbreviated in "sector so and so" and "outcome represented by the sector so and so" in "outcome so and so". These terminological licenses are not affected by any reasonable risk of interpretative confusions.

6.3.1.1. Incidentally. I choose a circle to recall Venn's diagrams, yet, if the alternatives were numerous, dealing with very narrow sectors would be an annoying practice; then a more general convention could represent $k°$ with a rectangle of convenient size, partitioned in the necessary number of sub-rectangles.

6.3.2. The second basic idea of ® is representing a piece of information h (for the moment h can indifferently be an acquirement or an hypothesis) by shading the sectors respectively precluded. Each of the 2^n different pieces of information concerning a possibility space of n alternatives can thus be unequivocally represented. For instance the piece of information that a not better specified outcome is B is represented by shading the sector 1 (obviously if an unspecified outcome is *B*, every sequence is possible except *AAA*); analogously the piece of information

(6.i) the first outcome is *B*

is represented by shading the sectors 1, 2, 3 and 4 because, of course, (6.i) precludes the four possible sequences whose first outcome is *A*.

I call "virgin" any non shaded sector and "virgin field" the union of the virgin sectors. So, for instance, the virgin field representing (6.i) is the lower semicircle.

The intensionality of ® results from the above explained ideas. Since its space does not represent sets of individuals, but pieces of information, any increment of information is represented by an increment of the shaded field, therefore by a decrement of the virgin field. In this sense the virgin field represents the residual margin of uncertainty.

6.3.3. Exactly in order to comply with this approach the third basic idea of ® is representing any alternative by a sector whose area depends on the characteristics of the represented universe, that is, roughly speaking, on the probability of the respective alternative. I make myself clearer. Until now I reasoned under a tacit assumption: the equiprobability of the various outcomes (here is the reason why I specified that the coin is well-balanced). And actually Figure 6.1 is suited to represent also different operative contexts, provided that the octo-partition of the respective possibility spaces be uniform. For instance, so that Figure 6.1 can correctly represent the results of a well balanced mini-roulette with eight numbers, the condition must hold that the eight boxes have the same size. A well balanced mini-roulette where the numbers 1 and 2 have a box of 80°, the numbers 3, 4, 5 and 6 have a box of 40°, the numbers 7 and 8 have a box of 20°, finds its correct (even iconic) representation in Figure 6.2.

6.4. I call "measure (of *h* under *k*°)", and I symbolize by

(6.ii) $\mu_{k°}(h)$

the quantity represented by the area of the *h*&*k*°-virgin field. Of course to speak of a triple $\langle k°, h, \mu\rangle$ would be a more canonical (but, in my opinion, an intuitively less adequate) expressive way. So, for instance, with reference to an octo-partitioned mini roulette, the measure of

(6.iii) the outcome is <5

is represented by shading the inferior semicircle if we refer to the *k*° of Figure 6.1, while it is represented by shading the last third of the circle if we refer to the *k*° of Figure 6.2; in fact as (6.iii) precludes the alternatives 5, 6, 7 and 8, it is represented by shading such sectors, thus leaving virgin the sectors 1, 2, 3 and 4.

The criteria ruling the assignation of a measure to the various alternatives of a possibility space will be studied in due course; here I only introduce the notion, emphasizing its intrinsically relational character. In fact the above example shows that the measure of (6.iii) depends also on *k*° (that is, in the case of the mini-roulette, also on the sizes of its eight boxes).

I advance that uniform partitions will be privileged; yet I underline that this agreement is not an artful way for intruding a sort of equiprobability; it is only a way for avoiding mathematic and geometric complications of no theoretical moment (a uniform partition is anyhow a partition).

6.4.1. Precisely as (6.ii) symbolizes the measure of the piece of information *h* under the statute *k*°,

(6.iv) $\mu_{k°\&h_1}(h_2\& h_3)$

symbolizes the measure of the piece of information $h_2\&h_3$ under the statute $k°\&h_1$ that is under the statute increased by adding the piece of information h_1 to *k*°. For instance, with reference to the *k*° of our mini-roulette, once h_i precludes the even numbers, h_2 is ~1 and h_2 is <5, (6.iv) is the measure of the outcome 3.

6.4.2. Obviously an actual measure corresponds to (6.ii) only on condition (sufficiency condition) that *k*° configure an actual possibility space. In other words we can say that an expression like (6.ii) succeeds in defining an actual measure only if the sufficiency condition is satisfied.

This condition is nothing but the intensional counter-part of the presupposition upon which the set-theoretic approach to probability is based; in fact (Hayek 2003 §1, Howson and Urbach 2006 §2.a}) such an approach presupposes a space of possibilities Ω (also: a universal set, a class of elementary events) and refers the probability of an event to the members of F, where F is the field of Ω-subsets.

6.4.2.1. If the measure of an *h* is actually defined with regards to a basic statute *k*°, then it is also defined with regards to a statute $k=k°\&h_1$; in fact if the information *k*° satisfies the sufficiency condition for *h*, the information obtained by adding h_1 to the previous one configures either the same possibility space ($h_1=\emptyset$) or a more punctual possibility space (that is a possibility space whose alternatives are a subset of the previous and less punctual one). Therefore while any increment of the basic statute does avoid the risk of losing sufficiency, a decrement does not.

For instance, the possibility of assigning a measure to the outcome 5 with reference to the *k*° of our mini-roulette entails the possibility of assigning a measure to the same outcome 5 with reference to the aforementioned statute $k°\&h_1$ telling us, say, that even outcomes are precluded, but it does not entail the same possibility if we forget the exact numbers of outcomes in such a mini-roulette.

6.4.3. Normally, increasing a quantity is increasing its measure. Therefore, since here we are dealing with pieces of information and the quantity of information $h_1\&h_2$ is equal ($h_2=\varnothing$) or greater than h_1, we might be induced to think that $\mu_k(h_1\&h_2)$ is equal or greater than $\mu_k(h_1)$. On the contrary we shall see even formally that $\mu_k(h_1\&h_2)$ is equal ($h_2=\varnothing$) or less than $\mu_k(h_1)$. This conclusion is confirmed by diagrammatical evidence; in fact the virgin field corresponding to $\mu_k(h_1\& h_2)$ is equal or less than the virgin field corresponding to $\mu_k(h_1)$. Let me insist: μ measures the residual uncertainty. In this sense calling μ "measure" is a potentially misleading choice. Yet such a choice is constrained by the current terminology; in fact (§7.12) the above defined notion (owing to the role it plays in the definition of probability) is strictly linked with the notion Waismann, Carnap and followers name "measure".

6.5. Indeed, once its rudiments have been understood, ® is a rather intuitive representation. Nevertheless some comments are suitable about the representation of the two connectives (conjunction and negation) through which we can create new pieces of information.

6.6. The conjunction $h_1\&h_2$ is represented by a shaded field whose sectors are the ones shaded by h_1 or by h_2. Yet a fundamental ambiguity arises. For instance, once assumed that h_1 precludes the last four sectors and h_2 precludes the sectors 1 and 8, is their conjunction represented by Figure 6.3

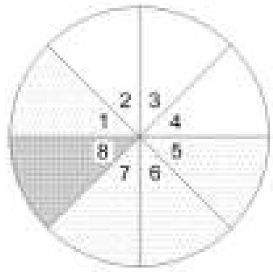

Figure 6.3

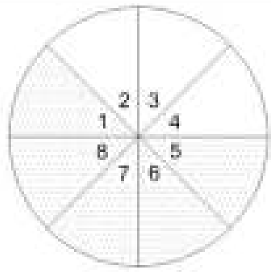

Figure 6.4

or by Figure 6.4? In other words: are there many or only one degree of shading?

My answer is that assuming (as in §6.1) the opposition between *known* and *unknown* without considering intermediate cognitive situations is assuming an idempotent logic where the reliability of any acquirement cannot be strengthened by iteration. Therefore Figure 6.4 is the right representation.

In other words. Once we assume that a piece of information h belongs to a statute k, no assumption is possible according to which, so to say, h super-belongs to k.

6.6.1. On these grounds implication is defined through conjunction; in fact

(6.v) $(h_1 \supset h_2)$ iff $(h_1\& h_2 = h_1)$

is immediately legitimated in ®.

Until (6.v) is considered as a strictly formal definition, it is a sovereign intervention which as such does not require any legitimation, but as soon as we relate *⊃* and *implication* the sovereignty fails. In fact defining formally a symbol that previously belongs to the current lexicon with a universally accepted meaning is a risky procedure, because it might facilitate argumentative abuses, that is undue inferences implicitly based on the universally accepted meaning. Yet, enucleating and emphasizing such a risk is already a good manner of keeping it under control. Moreover, in our case, no undue inferences are derivable, since it seems to me evident that in an idempotent logic if h_1 implies h_2, then adding h_2 to h_1 does not enhance the same h_1, and vice versa if adding h_2 to h_1 does not enhance h_1, this means that knowing h_1 is knowing h_2 too, therefore that the same h_1 implies h_2. For instance, with reference to our mini-roulette, if we know that only odd outcomes are possible (shading of the even sectors), to learn that an outcome is not 4 (shading of such a sector) does not increase what we already know. Therefore (6.v) complies with the usual meaning of "⊃".

If $h_1 \supset h_2$, I say that h_1 is an expansion of h_2 and h_2 is a restriction of h_1.

6.7. By

$$|h|$$

I mean the dilemma concerning the pair of opposite pieces of information h and $\sim h$ (§2.9.1). A dilemma is k-decidable iff k allows to infer which of its horn is true and consequently which is false. Therefore the decidability can indifferently be referred to any of the two opposite hypotheses.

6.8. The negation $\sim h$ is represented by a shaded field whose sectors are exactly the virgin sectors of h. That is: two opposite pieces of information are represented by complementary shadings. It follows that incoherence is represented by a wholly shaded circle; and actually incoherence is an excess of information (precluding every $k°$-compatible alternative, therefore contradicting the basic previous assumption). By

$$h\&\sim h = \perp$$

and

$$\sim (h\&\sim h) = \emptyset$$

I introduce the abbreviative symbols "$\perp$" (incoherence, excess of information) and "$\emptyset$" (tautology, null information). Therefore $\perp$ and $\emptyset$ are respectively represented by a totally shaded and by a totally virgin circle.

6.9. The maximal coherent information regarding a possibility space is represented by shading all the sectors but one; in fact such a virgin sector, telling us exactly the actual alternative, crowns our way to a complete knowledge. In this sense I call "exhaustive" a statute leaving only one free alternative and "k-exhaustive" a piece of information h such that $k\&h$ is exhaustive. For instance, in its 'pitch-and-toss-interpretation' of Figure 6.1, if we know (h_1) that the second outcome was B, to learn (h_2) that no pair of consecutive tosses gave the same outcome is a $k°\&h_1$-exhaustive acquirement, since it allows us to infer that the sequence is ABA. Yet the same acquirement is not $k°$-exhaustive, since not to know that the second outcome was B, admits also BAB as a $k°$-compatible sequence.

6.10. Any propositional connective can be formulated in terms of conjunctions and negations; therefore, once we know how to represent conjunctions and negations, we also know how to represent any propositional connection. For instance, let me consider informally the inclusive (OR), the exclusive (NAND) and the partitive or disequivalent (XOR) disjunction (their formal introduction in § 7.10.1); then Figure 6.5, Figure 6.6 and Figure 6.7

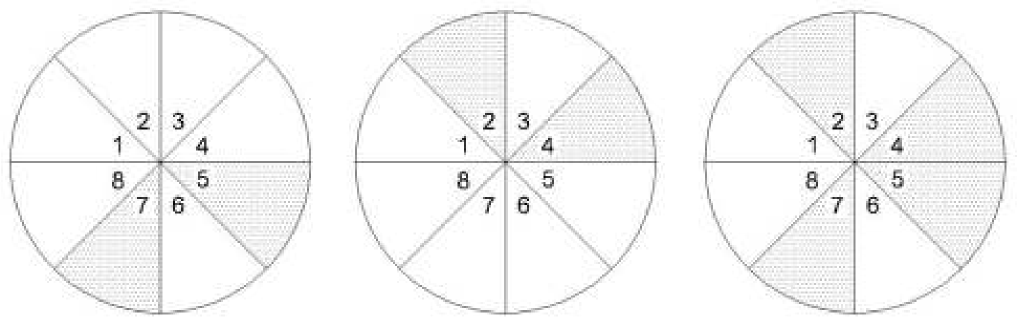

Figure 6.5 Figure 6.6 Figure 6.7

represent respectively the three disjunctions of (6.iii) and of
(6.vi) the outcome is even
with reference to our mini-roulette. The three figures have been obtained through the strictly graphic procedures dictated by the definition of the disjunction under contingent scrutiny, that is, respectively,
- OR: shading the sectors shaded both in the representation of (6.iii) and of (6.vi),
- NAND: shading the virgin sectors both in the representation of (6.iii) and of (6.vi),
- XOR: shading all the sectors shaded in the two preceding figures.

Yet the same figures can also be obtained through simply argumentative procedures. For instance (Figure 6.5) the inclusive disjunction of (6.iii) and (6.vi) precludes only the eventuality of an odd and >4 outcome, because such an outcome would contradict both 6.iii) and (6.vi), therefore it would falsify the same disjunction. And so on.

6.11. Since in next chapters these considerations will be retaken and deepened, I propose a particularly flexible example. A tract of a rail is partitioned in eight consecutive segments of different colours, and an automatic spring mechanism pulls a slider on the rail; the slider moves any time from the same starting point and its final position is exactly determined by a lubber line, so that the eight alternatives correspond to the eight segments where the slider can stop. Although it would be easy to sketch contexts where the final position depends on many parameters (variable friction, non horizontal altimetry et cetera) I assume that the resistances to the motion are the same in every point of the tract, and thus that the distance is the only parameter (monoparametric context). Of course the final position of the slider depends on the impulse I transmitted by the spring, and this impulse ranges from an $\mathcal{I}_{min}$ to an $\mathcal{I}_{max}$ according to which the slider stops respectively at the beginning or at the end of the tract.

First of all the flexibility of the example allows to see Figure 6.1 and 6.2 as concerning simply two different partitions of the tract (the area of every sector is directly proportional to the length of the segment it represents).

Such a flexibility could also allow the introduction of new distinctions based on further connotations. For instance a distinction based on the hypothesis that every segment of the tract is partitioned in three sub-segments, each of them identified by a nuance of the respective colour would increase the respective possibility space and could be represented in the intuitive way (the partition of each sector in three subsectors). Nevertheless, since an octo-partition of the circle is already sufficient to support adequate examples and since a non-uniform assignation would mean overcharging the exposition with arithmetical complications of no theoretical moment (let me repeat: uniformity too, after all, is an assignation of measures, and its peculiarity does not influence the following discourse), the statute $k°$ represented in Figure 6.1 will be privileged. Of course once every sector is considered as an elementary entity, every piece of information referring to further distinctions is of no moment. For instance

the slider stopped in the second half of the sector 3

does not increase

the slider stopped in sector 3

since under $k°$ the specification adduced by “second half of” is meaningless when referred to a segment of the tract.

6.12. The above representation presupposes the assignation of a precise measure to the various alternatives. Of course any partition where the virgin (free) field is composed by more than one sector represents an uncertainty (the certainty is represented by a monosectorial virgin field whose only sector represents exactly the actual alternative). Yet a ‘meta-uncertainty’ may concern the same assignation of a measure to some alternative. If we are studying the three tosses sequence of a well balanced coin, Figure 6.1 is undoubtedly the right one; but not all contexts are analytically quantifiable. For instance, what about the possible results of a foot-race with eight competitors? Of course we can assign to each of them a measure inspired by our previous information on his chances, so in general obtaining a non-uniform diagram like, say, Figure 6.2. This notwithstanding any single assignation might suffer a margin of ambiguity (different bookmakers might propose different rates). A situation like this could be represented by fuzzy sectors, that is by replacing the various radii which separate the various sectors with sub-sectors more or less narrow according to the margin of ambiguity in the assignation of the respective measure. Anyway I neglect such a topic because my present task is only to sketch the theoretical frame of a representation, not to enter into details of its possible refinements.

6.13. Let $k=k°\&k'$ be the statute concerning the possibility space under scrutiny. According to the above explained technique (henceforth I call it “by shading”) we shade the k'-sectors of a circle partitioned in conformity with $k°$. But, since shading a sector is precluding it (that is cancelling it from the possible alternatives), another technique (I call it “by re-partitioning”) is performable: drawing a new and completely virgin circle whose sectors are only the virgin sectors of the previous one. For instance, once translated into the re-partitioning technique, the situation represented in Figure 6.7 leads to a uniformly tetrapartitioned circle (uniformly because the four virgin sectors of Figure 6.7 have the same area and the areal ratios of the virgin sectors, obviously, must be conserved).

The re-partitioning technique, follows from interpreting the shadings of this technique as a radical elimination of the respective alternatives from the possibility space (so entailing a re-distribution of the resulting vigin field). Yet the most general approach concerns acquirements that, far from forbidding the radical elimination of an alternative, increase or decrease its respective assignation (that is the area of the sector representing such an alternative). This topic will be faced in §13.6.2.

6.14. The incidental note of §6.3.1.1 deserves a last remark.

I only know that A and B are among the (how many?) competitors of a foot race, and that their chances are the same. This situation, which cannot be represented through a circle because of its fixed area, can be represented through an open rectangle (where two equal sub-rectangles are part of a rectangle whose total area is unknown).

CHAPTER 7
AXIOMS

7.1. Here I formalize the informational approach through a propaedeutic system of logical axioms. The reasons why I speak of a propaedeutic system will be explained in §8.7.1. Obviously the variables *"h", "h_1"* et cetera range over pieces of information.

The axioms are

AX1 IDENTITY	if h then h
AX2 REPLACEMENT	if h_1 and if $h_1 = h_2$ then h_2
AX3 ASSOCIABILITY	if h_1, and if h_2 then $h_1 \& h_2$
AX4 COMMUTABILITY	if $h_1 \& h_2$, then $h_2 \& h_1$
AX5 RESTRICTION	if $h_1 \& h_2$, then h_1
AX6 COHERENCE	if $h_1 \& {\sim}h_1$, then h_2
AX7 COMPLEMENTARINESS	if $h_1 \& h_2 = h_1 \& {\sim}h_1$ then $h_1 \& {\sim}h_2 = h_1$

and could be re-proposed in the well known fractional notation where for instance AX2 becomes

$$\frac{h_1 \qquad h_1 = h_2}{h_2}$$

et cetera. I preferred "if then" only for minute typographical convenience.

7.1.1. The above axioms draw an idempotent logic (§1.3). In fact

Theor1. If h, then $h \& h$ and if $h \& h$, then h.
Proof, By AX1 and AX3, if h then $h \& h$; by AX5, if $h \& h$, then h.

Obviously the theorem of idempotence concerns pieces of information. For instance it states that, if we know that 2<3, we can infer that 2<3 and 2<3; therefore, since reciprocally (AX5) if we know that 2<3 and 2<3, we can infer that 2<3, to know that 2<3 and 2<3 is nothing more and nothing less than to know that 2<3 (Theor4 below).

7.2. Some comments are opportune in order to show that the usual interpretation of symbols satisfies the admissibility criterion.

7.2.1. AX1 is obvious: any piece of information can be inferred from its assumption.

7.2.2. AX2 rules the substitution of identity. Indeed "substitution of identity" is a patent oxymoron, because in the meaning of "substitution" there is a component of diversity quite incompatible with the very meaning of "identity" (scholasticism taught: *si duo idem faciunt, non est idem*); this notwithstanding I respect the current terminology. Anyhow the topic will be better analyzed in Chapter 8.

7.2.2.1. Besides the substitution of identity, current theorizations list *Modus Ponens* as a further inference rule. Here, by the definition (6.vi), it is a theorem. In fact

Theor2. If h_1 and if $h_1 = h_1 \& h_2$, then h_2.
Proof. By AX2 we get $h_1 \& h_2$, then, by AX5, h_2.

Of course if "h_1" and "h_2" were variables over sentences,

$$h_1 = h_1 \& h_2$$

would be an absurdity.

7.2.3. The admissibility of AX3 follows immediately from the same meaning of "and", that is from the same *conjunction* (when referred to pieces of information). The repetition of "if" says just that the two acquirements are singularly considered.

7.2.4. At first sight a superficial objection might suggest some perplexity about the admissibility of AX4, that is about the commutative property of conjunction. For instance

(7.i) Ava took a lover and Ava's husband abandoned her

and

(7.ii) Ava's husband abandoned her and Ava took a lover

are obtained by commutating the same two atomic statements, yet (7.i) and (7.ii) adduce two different pieces of information, otherwise it would be unexplainable why the respective lawyers are quarrelling about (7.i) and (7.ii).