# APPENDIX TO CHAPTER 18
# SET THEORIES

A18.1. Contemporary set theories struggle in well known difficulties. Quine (1963, Preface) writes: *the axiomatic systems of set theory ... are largely incompatible ... no one of them clearly deserves to be singled out as standard ... intuition here is bankrupt*. A rather disconcerting conclusion, indeed, not only because the notion of a set seems highly intuitive. but also because a crucial question imposes: what ought the choice of axioms be dictated by, if not by our intuition?

A18.2. The proposed solution of Russell's paradox (§18.6.2), bans many absurd theorems of the current set theories (first among them the mentioned theorem according to which *{x: x=x}* is empty). Another source of paradoxes is represented by arguments depending on notions like *all*, *the greatest* et cetera. The following suggestion might be useful.

A18.3. A set *z* is well defined over a universe of reference Ω iff for every individual *x* of Ω the definition allows us to decide whether *x* is a member of *z*.

A set can be defined either intensionally (by establishing the characteristic an individual must possess in order to be a member of the set under formation) or extensionally (by listing its members). Since extensional definition are quite unproblematic, henceforth only intensional definitions are considered.

A18.4. In compliance with Cantor's original position, a set is *a collection of entities of any sort* (Suppes 1972, footnote p.1); therefore a collection of sets, too. The peculiarity characterizing the sets of sets is that while a set of apples is not an apple, a set of sets is a set. A basic distinction opposes the sets of sets such that

(A18.i) *P{x: Px}*

(let me call them "open") to the sets of sets such that

(A18.ii) *~P{x: Px}*

(let me call them "closed"). Referring such a distinction to sets of sets is a necessary condition to assure the properness of (A18.i), where the same predicate is ascribed to *x* and to *{x: Px}*.

The opposition between open and closed sets can be directly extrapolated to the respective conditions of membership (that is to their intensions).

The successor *m'* of an open set *m* is the set whose members are the same *m* and the members of *m*.

A18.5. Obviously the set of those individuals that are *P*, besides depending on *P*, depends on the universe of reference. Shoenfield (in Barwise 1977, §2) writes: *When we are forming a set z ... we do not yet have the object z.* I agree totally: no set can pre-exist to its birth. This means that, given a basic universe of reference *Ω°*, when we are dealing with an open set *m*, the universe resulting from the union of *Ω°* and *m* is a new universe *Ω'* different from *Ω°* (and so on). In this sense

(A18.iii) *{x: Px}*

is a correct notation only if the set is closed, that is iff (18.ii) holds. Otherwise (18.iii) is an ambiguous notation where it is not specified if we are referring to *Ω°* or to its successor *Ω'* (and so on); in this sense open sets entail diachronically mutable universes of reference. And quantifying over a diachronically mutable universe without specifying the moment our quantification refers to is introducing a heavy lack of information; for instance an expression like

No *x* exists such that …..

opens spontaneously the way to the question: **<u>when</u>** does not it exist? Realizing that what does not exist at *t°* (that is with reference to *Ω°*) may exist at *t'* (that is with reference to *Ω'*) is the essential acquirement in order to avoid an incumbent risk of preposterousness. And it is easy to verify that all impasses concern open sets; for instance the membership requisite for the set of all sets is simply to be a set.

A18.5.1. For the sake of concision and in conformity with previous assumptions I only consider increasing universes. So if something does not exist at *t*, then it cannot exist in any preceding moment, and if something does exist at *t*, then it does exist in any following moment.

A18.6. As far as I know a meticulous respect of our intuition leads to a sound and unproblematic diachronic set theory, that is to a theory whose basic rule is avoiding any preposterousness by exacting the chronologic specification of every open set. Therefore my suggestion is simple: diachronizing formally the various expressions through qualified indexes. So for instance

$\{x: Px\}''$

names the set of those $x$ that at $t''$ are $P$,

$\sim\exists' x(Px)$

says that at $t'$ no $x$ exists such that $Px$; analogously

$\exists!^{t} x(Px)$

abbreviates

$\exists^{t} x(Px) \ \& \sim\exists^{t} y(y \neq x \ \& \ Py)$

and so on. Under this convention, for instance,

$\exists!^{t+1} x(Px) \ \& \sim\exists^{t} x(Px)$

can be interpreted as the birth at $t+1$ of exactly one $x$ such that $Px$.

A18.6-1. An immediate example of the advantages offered by such a diachronization is supplied by

(A18.iv) $\exists^{t+1} y \ \forall^{t} x(x \in y \leftrightarrow Px)$

representing the axiomatic scheme of abstraction in its new formulation. And actually (A18.iv) precludes any paradox: for instance,

$\{x: x \notin x\}' \in \{x: x \notin x\}'' \leftrightarrow \{x: x \notin x\}' \notin \{x: x \notin x\}'$

far from being self-contradictory, represents exactly our intuitive convincement.

A18.6.2. Let me insist. A notation like (A18.iii) is admissible for closed sets because, written concisely,

$(\sim P\{x: Px\}^{\circ}) \supset (\{x: Px\}^{\circ} = \{x: Px\}')$

that is because the formation of $(\{x: Px\}^{\circ}$ does not increase the set of sets satisfying the same condition of membership. On the other hand (A18.iii) is a notation to reject if the set is open because

$(P\{x: Px\}^{\circ}) \supset (\{x: Px\}^{\circ} \neq \{x: Px\}')$

and the lacking chronologic index then makes (A18.iii) a referentially ambiguous expression.

A18.7. I suppose that the well known opposition between sets and (proper) classes might be reduced to the opposition between open and closed sets. Indeed the current notion of classes is not well defined. On this subject Suppes (1972. §2-6) acknowledges that *classes appear rather bizarre from the standpoint of naive, intuitive set theory.* (cf. also Lake, 1974 p.415). And anyhow some basic incompatibilities among the intuitions of the most celebrated theorizers of the distinction between classes and sets are undeniable. For instance the question

may a class be identified with the set having the same members?

would be answered negatively by Bernays and affirmatively by Goedel (1958, footnote 5). Of course somebody could object that intuition is an optional in an axiomatic theory; but I could not only evoke Kleene (*An intuitive mathematics is necessary even to define the formal mathematics ... the ultimate appeal... must be to the meaning and evidence rather than to any set of conventional rules*); I could also re-propose a heavy reply based on the admissibility criterion: what ought the axioms be legitimated by, if not by intuition?

In my opinion there is no need to assume the notion of classes among primitives. Nelson Goodman (1943, p. 107) writes

> A given idea *A* need to be left as primitive in a system only so long as we have discovered between *A* and the other primitives no relationship intimate enough to permit defining *A* in terms of them

and I cannot but agree; yet in my opinion classes are open sets. And an open set can be the member of another set under the previous fixation of the moment of reference since, otherwise, the lack of a chronologic index would involve us in a referential ambiguity (in a lack of information). Could we correctly ask how many inhabitants had Rome without specifying the moment of reference? And are we compelled to specify the moment of reference when we ask how many edges has a cube?

Here too preposterousness is the root of the impasses.

A18.8. Preposterousness is an essential factor in diagonal procedures and Berry's paradox is a symptomatic example. Shortly, such a paradox arises as soon as we realize that

(A18.v) the smallest number which cannot be succinctly defined

is succinctly defined by (A18.v). Though its solution can be led back to the argument proposed in Chapter 18, let me face it through a peculiar analysis.

A number can be defined on the grounds of some specific connotation. Formally we must only appeal to connotations derived from primitive notions; for instance 3 is the successor of the successor of the successor of 0. Yet informally we can appeal to any connotation resulting from the statute of reference; for instance 3 is the number of Graces or of triumviri et cetera. Thus we list a set *m* of definitions which, in its turn, can be assumed as evidence for new connotations. But if we appeal to a connotation like that, we are necessarily involved in an open context, and we then must avoid any confusion between the previous set *m* and, speaking trivially, its successor *m'* obtained by adding the definition we are performing to the same *m*. As such the smallest number which, with reference to *m* cannot be succinctly defined, may be succinctly defined with reference to *m'*. In other and more general words. Any quantification must be referred to the domain of the variable under quantification; and to speak of the least number which in *m* is not defined in a certain way is to say that actually in *m* there is no definition such that et cetera, therefore it is a quantification; yet of course a correct quantification over *m*, may be incorrect over *m'*. In this sense the argument recalls Richard's original solution (I remind the reader Richard's *at the place it occupies, it has no meaning*); in this sense Berry's paradox too is born by a preposterous pseudologism.

A18.9. The informational approach, precisely because of the central role played by the knower, then by meanings (intensions, connotations, properties), rejects the exasperated extensionalism of the current theorizations of logic. Identifying an *n*-place relation with the respective set of *n*-uples is violating our most deep-rooted cognitive mechanisms. When I say that the matrimonial relation is dangerous I am not at all saying that the set of conjugal couples is dangerous; I am speaking of a link, not of a collection. Since two sets having the same members are the same set and since the set of individuals with a hearth is the set of individuals with a liver, how could a strictly extensional approach recognize their strong distinctive factor? The intensional step (inquiring into a characteristic) is very often necessary to ascertain that some individual is a member of some set. Therefore no valid theoretical approach can ignore that our mind avails itself of intensions not less than of extensions. I am not fighting against extensionalism, I am fighting against its pretension to be the only approach, a pretension which seems to me a desperate attempt to ennoble through a strict formalism a truly stone age metaphysical perspective.

A18.10. A last consideration entailing a metaphysical compromise focuses on the credentials on whose grounds the same relation $\in$ of membership is assumed as a primitive notion. Reality is unitary. Only for gnosiologic convenience we partition it in a multiplicity of individuals (and the necessary net of relations is the price to pay for restoring the unitarity). Of course, usually, our partitions are firmly suggested (but not logically imposed) by objective physical discontinuities (we see a rider and his horse as separate individuals, Aztecs saw *conquistadores* on horseback as single monsters, nobody sees the rider and half horse as a single individual). A set is nothing but an individual resulting from the assumption *ut unum* of a collection, that is of some otherwise autonomous individuals resulting from a previous partition; and such an assumption *ut unum*, roughly, is a sort of conjunction. In this sense $\in$ can be reduced to a relation of identity between the member we are speaking of and one of the conjoined individuals (that is to a disjunction among identities between individuals). This viewpoint might also shed light on the hardly debated relation between individuals and singular sets.

Anyhow I emphasize that mine is a mere suggestion.