INTRODUCTION

Felix qui potuit rerum cognoscere causas

Prolix introductions are often neglected. An excellent reason to be concise.
Many a symptom tells us that contemporary Logic is seriously ill. What credit could the common sense recognize to someone who assured the possibility to dissect an orange in such a way that with its pieces two oranges identical to the former could be recomposed? Neither Banach-Tarski’s and Hausdorff’s theorems (Skolem 1962, Lecture 11, or, more in details, Jech in Barwise 1977) are the sole manifestations of a severe disease. The situation of uncertainty determined by the fact that logical paradoxes are still waiting for a general and definitive solution compels the orthodoxy to nearly unbelievable contortions in order to save a merely apparent coherence.

Some examples. What about the theorem of the classical set theory according to which (Suppes, 1972 Theor.50) the set of individuals identical to themselves is empty? And what about the total elimination of variables (Schoenfinkel in V. Heijenoort 1967)? What about Lowenheim’s Theorem? What about the truth functional analysis of conditionals? What about the theorization of truth?

Nowadays Logic is a freakish and esoteric doctrine far from our common sense, and the crumbling zones are too numerous to hope that the vice be superficial. Problems cannot be solved until we do not get out of the approach by which they have been born, says an old adage. Actually a radically new approach is needed.

For an unknown author, an almost infallible way to fall into ridicule is to begin by declaring that his results are important, My dilemma is then this: to lie or to face the charge of intellectual autolesionism? Someone says: My aim? To show the fly the way-out from the trap, and suddenly a chill casts over the gathering; yet the aphorism is not mine (Wittgenstein, 1953 p.309). It is a question of previous authoritativeness and mine, probably, entails a general inattention. Nevermind: the success is nearly always wrong, if Schopenauer is right. I am not so naive to think that history demonstrates the unavoidable success of meritorious works. History demonstrates only the unavoidable success of the meritorious successful works. What do we know about the rest? A bad destiny, yet not the worst: chronicles narrate that Hyppasus of Metaponto was put to death for having made public the existence of incommensurable quantities (hence my temptation to proceed apocryphally).

On the other hand already Locke (probably inspired by my contingent situation) realized that a simple and private writer cannot avoid being censured when pursuing the truth through an autonomous path (too many quotations? they are simply an awkward attempt to mask frightening cultural gaps).

Then to write with a serene irony, once the expositive clearness is assured. A really courageous aim, since an obscure text is the strongest defence against the Wildian terror of not being misunderstood.

Indeed an altruistic reason too induces me to publish this work: to put a new and fecund viewpoint at public disposal. Yet the main reason is egoistic: to establish a paternity. I confess that I would be proud to be considered as a great thinker; also because of the high social qualification which traditionally is recognized to the category; he is a stupid, he thinks too much, King Arthur admonished.

So to say roughly, the contemporary logic is crushed down by an intrinsically inadequate bidimensional approach: on one side the sign, on the other side the referent. But such an approach neglects the crucial protagonist of any reasoning, that is the information, the meaning, the mind. I agree that “information”, “meaning”, “mind” express difficult and dangerous notions, but to look elsewhere when they appear on the stage is a loosing ostrich strategy. The primigenial logic that allowed the animal evolution until the homo sapiens sapiens, cannot be a logic of the sign, since the sign appears millions of years later: it must be a logic of information, since information does exist wherever an even primigenial knower exists. No doubt that the sign and the referent too have an irrenunciable place in every realistic gnosiology, but to assume them as sole actors is like to reduce a cruise to the embarking and the landing.
As soon as a third dimension is granted to logic, its problems find a sequence of harmonious solutions. The evidences are too many to represent mere concomitances; only when the path is the right one the lifting of the fog reveals a landscape corresponding to the map we are following. In this sense I firmly believe that these same ideas, perhaps better proposed by other authors, but anyhow these same ideas, sooner or later, will enter into the history of Logic.

I spoke with absolute sincerity. Some smart spirit (Russell?) remarked that a reasonable private income is an indispensable requisite to practise the sincerity. Someone else (who?), not less smartly, remarked that to practise the sincerity is a rather dangerous behaviour if it is not coupled with a massive dose of dullness. Discouraging remarks for him who, like me, possesses a reasonable private income, shrinks from any danger and nevertheless thinks to be in the best conditions to practise the sincerity.
1.1. In my opinion, the very consequential mistake affecting the current logic hides, so to say, in its constitutional flatness. At first sight, to by-pass meanings (fuzzy and elusive entities) in order to deal with signs (well determined and tangible objects), seems a quite advantageous choice. Yet such a decision presupposes erroneously that the meaning-stage is by-passable as an unessential intermediate landing. A theory which puts on the one hand things, properties, relations, and on the other hand nouns, adjectives, verbs, neglects the real protagonist of whatever logos, that is the knower (the intelligent organism) whose presence is absolutely necessary in order to settle a contact between the sphere of referents and the sphere of signs. Logic does not rule the world; it does rule the knowledge of the world: what space could logic have in a galaxy without any gnosiology? But where is the gnosiologic dimension in the current theorizations? An approach reducing a tridimensional matter to bidimensionality is intrinsically inadequate, and since the neglected third dimension is just the informational one, only a logic of information can overcome this inadequateness.

1.1.1. To avoid misunderstandings, I am not claiming that the current theorizations are sterile; they are partial and reductive like the visit of a doctor who were to look at his patient exclusively through a little mirror. I claim that the only logic compatible with our common sense must be grounded on the concept of information, and that such a concept, wherever a linguistic component occurs in the process, is nothing but the concept of meaning, an indispensable trait d’union between sign and referent.

I recognize that to speak of meanings ex abrupto is a license. I agree with Chomsky where he says (I quote by memory) that the problem of meaning is huge and confused; modestly, my detailed notes on a semantic theory, though still in progress, already occupy hundreds of pages. For the moment I acritically appeal to the reader’s intuitive meaning of “meaning”, provided he agrees that pieces of information (then meanings) are a question of neurons.

1.2. Advanced natural languages possess splendid faculties of self-improvement. Nevertheless, obviously, the first steps of the improvements can only move from the not yet improved situation. So I feel myself like an electrician who, called to renovate a plant, uses the old one to illuminate his work. Anyhow the renovation that I will agree on in this book is limited to fill up the gaps whose filling up is necessary to achieve the purposed results.

1.3. The concept of information is polyvalent (information about something, information as something et cetera). Here I do not analyze it; I simply specify with a propaedeutic aim what I mean when I speak of information. And since I think that many potential misunderstandings follow from a too ambitious schematization of highly complex universes of reference, for the moment I focalise the outcomes of tossing a standard die.

The concept of information is strictly linked to the concept of knowledge or, even better, to the concept of ignorance. A (non-null) piece of information (concerning a certain phenomenon) is any acquirement reducing our ignorance about such a phenomenon. Let me toss the die and let me suppose that we attain only to see the centre of the resulting face, so ascertaining the presence of a dot. The possibility space $\Omega$ (that is the set of all alternative outcomes) is $\Omega = \{1,2,3,4,5,6\}$. Then our acquirement (a dot in the center of the face) is a (non-null) piece of information (concerning the toss under scrutiny) only because it allows to exclude 2, 4 and 6 from $\Omega$. In this sense a piece of information (concerning a certain possibility space) implies the reduction of the possible outcomes.

Of course the piece of information we can draw from a certain acquirement depends also on our previous knowledge. We continue referring to the above mentioned $\Omega$ but if in a standard die the dots of a face were the vertexes of the correspondent regular polygon, the same acquirement (a dot in the centre of the face) would exclude 3 and 5 too, so allowing us to infer that the outcome is 1. And as soon as the possibility space reduces itself to a single alternative, the outcomes cannot be further reduced (actually, once we know that the outcome is 1, under $\Omega$ no further information about such a toss can be acquired). The sight of the whole face could only confirm a piece of information we already know (the central dot is the only one). Just on these grounds I say that the logic of information must be idempotent (§7.1.1).

1.3.1. A pedantry. Above I wrote “under $\Omega$ no further information ... can be acquired” to specify that the many further pieces of information we can acquire once we know that the outcome is 1 (as for instance the position of the die on the green baize) do not concern the possibility space under scrutiny, since it classify the different outcomes as regards exclusively the number of dots marked on the six faces. And as soon as we refer to a more complex possibility space, the outcome 1 contains a plurality of sub-alternatives, so then further information is possible.
1.3.2. The possibility space approach could be replaced by a (Kripkean) possible worlds approach. Yet I strongly prefer the former because it seems to me that our effective gnosiologic life refers spontaneously to only one world and to acquirements able to reduce our ignorance about possible outcomes (in other words: the possible worlds are nothing but a fictitious way to deal with possibility spaces born by our ignorance about the real one).

1.4. If \( a \) is a state of affairs (for instance the presence of a dot in the centre of the face) concerning the universe of discourse \( \Omega \), the tossing of a standard die then \( k_{a,g,t} \) is the piece of information (the image, the neuronal pattern) the knower \( g \) possesses about \( a \) at the moment \( t \). By “\( \gamma_{g,t} \)” I generically indicate the cognitive relation (for instance the sight of the central dot) between \( a \) and \( k_{a,g,t} \) (\( \gamma_{g,t} = \gamma_{g,a} \)). The \( \Omega -g-t \)-statute is the union of the pieces of information about \( \Omega \) possessed by \( g \) at \( t \), that is, so written concisely, \( k_{\Omega,g,t} \) (for instance the knowledge about the dots marking the various faces of a cube, about the tossing procedures et cetera).

If we refer to an ideal knower \( ig \) (an auspicious acronym, indeed) the specifications “for \( g \) at \( t \)” can be omitted since, thanks to a hot line to God, an ideal statute is not subjected to chronologic or personal variations.

1.5. The concepts of information and of communication are strictly related; in fact a communication can be defined as a process of transferring information. Yet the concept of information is extremely broader, since the main channel to acquire information (the only channel, for millions of years) is (was) quite independent of any interpretation. While the concept of information implies the figure of a knower (or interpreter, or receiver) it does not imply the figure of a speaker (or sender). A sound reason to privilege the viewpoint of the interpreter.

1.5.1. In Fedro, Plato argues in favour of spoken languages and against written ones. Without discussing the topic, here I agree that, on the contrary, my discourse will mainly focalize written languages. Therefore expressions like “speech act”, “utterance” et cetera will be used without any intrinsic reference to the phonic dimension.

A systematic theorization of the matter will be proposed in a subsequent book specifically devoted to such a task. Here I only sketch a linguistic process of communication performed under the most simple conditions (so lies, mistakes of codification, noises, distrustful interpreters, emotional components et cetera are banned). In compliance with this frame

is schematically analysed as a tetradic relation: someone (the speaker) communicates something (a piece of information), to someone (the interpreter) by means of something other (the uttered expression \( e \)). Incidentally: personal note-books (or, where non-verbal communications are admitted, knots to handkerchiefs) show that speaker and interpreter are not necessarily distinct persons; anyhow, even where the speaker is also the interpreter the relation continues being tetradic, exactly as the division of a number by itself continues being a dyadic operation.

1.6. In order to speak of something we must use signs. Not only we can speak of the Koh-i-noor without exhibiting the diamond, but if we exhibit it, we are no longer speaking of it, we are exhibiting it.

Signs are intrinsically means to adduce conventional information (“to adduce” is just the technical term to express the semantic relation \( \sigma \) between a sign and the conventional piece of information it is the bearer of). Being the source of a \( \sigma \)-relation is the essential requisite in order to be a sign (without such a requisite, a sign would only be a mute object merely displaying itself as a blade of grass in a meadow). The \( \sigma \)-relation is necessarily mental: The linguistic faculty is just the mental faculty of stating conventional associations between arbitrary couples of informational nuclei. The best evidence is the sovereignty of the decision through which we can assume any object as a name of any other object; a sovereignty showed by the possibility of establishing such an assumption in complete darkness and silence and immobility.

1.6.1. Any language is a code; then any codification or decodification depends on the language we are referring to. Yet for the sake of concision (cf. also §1.9.2 below) I agree to make explicit such a reference only where the context could otherwise entail some ambiguity.

Let \( e \) be a sentence (of a language \( L \)). I say that (in \( L \)) \( e \) adduces the piece of information \( h_{g,t} \) iff in \( g \) at \( t \) a (mental) relation \( \sigma_{g,t} \) does exist such that \( \sigma_{g,t}(\gamma_{g,e}) = h_{g,t} \). I call generically “semantic” the relation \( \sigma \).

Of course the relativizations to \( g \) at \( t \) can be omitted where reference is made to the ideal knower.

1.7. The basic process of communication can be roughly sketched as follows

- the speaker \( g' \) intends to communicate a piece of information \( h_{g'} \)
- the speaker knows that \( h_{g} \) is adduced by \( e \) (linguistic codification)
- the speaker utters \( e \)
- the interpreter \( g'' \) reads \( e \)
- the interpreter knows that \( e \) adduces \( h_{g''} \) (linguistic decodification)
- the interpreter acquires \( h_{g''} \)
Thus a process of communication is partitioned in two stages. The enunciatory stage leads the speaker to find and to utter the expression adducing the piece of information to transmit, the interpretative stage leads the interpreter to draw the piece of information adduced by the expression he reads.

Some comments.

1.7.1. Of course the scheme above is oversimplified. In order to generalize it we ought, at least,
- introduce chronological indexes (for instance the moment \(g\) reads \(e\) may differ from the moment \(g'\) utters it)
- the speaker may adapt the enunciatory stage to the interpretative stage he presumes and vice versa

Yet these pedantries are of no moment in the present analysis whose only aim is emphasizing a clean distinction between signs (\(e\)) and pieces of information (\(h\)). Moreover a not less clean distinction is emphasized between pieces of information and referents. For instance the speaker may intend to communicate that it is raining quite independently of the actual meteorological situation.

1.7.1.1. The same sketch can be easily conformed to peculiar situations. For instance situations where the message to communicate concerns in its turn a meaning, or where the code is non-verbal (a non-comprehension communicated by a grimace instances both).

1.7.2. To speak of meanings is incompatible with an extensional approach. Indeed the severe war between extensionalists and intensionalists seems rather gratuitous to me. A bipartisan pantensionalist approach witnessing our actual and evolutionally tested gnostiologic procedures seems to me the best one. Just as the easiest way to analyse the opinion of a guard controlling the guests is extensional (the guard is called to ascertain whether the (name of the) person under scrutiny belongs to a list, quite independently of his/her underlying requisites), the easiest way to analyse my opinion about the little animal squatted beneath the bush is intensional (I think it is a rabbit because I saw that its physical connotations and its behaviour are characteristic of the species, surely not because I ascertained that it belongs to a set the majority of whose members are unknown to me).

1.7.3. The speech acts we shall deal with are declarative, yet also performative speech acts (such as questions, commands and so on) concern processes of communication, since anyhow they adduce some piece of information (a question informs the interlocutor(s) that an answer about a certain argument is wished, an order informs the interlocutor(s) that a certain behaviour is required and so on).

1.8. Usually, in a specific process of communication, an important role is played by what Nørretrander (1998) calls “exformation”, that is by the unexpressed body of knowledge the speaker and the interpreter share. It seems to me that the correct approach to the topic entails a clear distinction between
- the piece of information constituting the strict (content of the) message
- the piece of information which, on the ground of his previous statute, the interpreter can infer (I evoke the tossing of a die).

The example of the dots marking the various faces is enlightening. If the speaker utters

(1.i) the centre of the face is marked by a dot

the interpreter infers

(1.ii) the outcome is either 1 or 3 or 5

if he thinks that the faces are marked in the usual way, while he infers

(1.iii) the outcome is 1

if he thinks that the faces 3 and 5 are marked in the vertexes of the respective regular polygon. And of course the speaker can be perfectly aware that the piece of information he transmitted is not the mere (1.i) but, according to the real disposition of the dots, it is (1.ii) or (1.iii). The intriguing study of the various intents moving the interpreter is here forbidden by the above agreed restrictive conditions.

Once these considerations are extrapolated from the analytically classifiable possibility space of a die (then from a set of analytically foreseeable inferences) to the extremely complex possibility space of our old world, the concept of exformation tends to assume a less sharp profile.

1.9. No care is wasted if it can preserve the interpretation of a text from ambiguities. The basic requisite is simple: not to entrust the same sign with different semantic tasks. That is, briefly: no homonymy bearer (no word adducing more meanings). Homonymy is our irreducible enemy, since as soon as an informational non-identity is hidden behind a signic identity, a potential source of incoherence is introduced in the process of communication. And the less clear our ideas about the various pieces of information adduced by the sign are, the more insidious the homonymy is. In this sense the worst one is an autonymic homonymy, where the same sign is one of its possible referents. In the final chapter an astonishing example of the dangers born by an analogous context will be studied in detail.
1.9.1. A less hasty analysis of homonymy bearers suggests a specification. While “sentence” is used to mean a syntactic entity (that is a concatenation of words respecting the well formation rules) and “proposition” is used to mean a semantic entity (that is the piece of information adduced by a sentence), “message” will be used to mean either a sentence or a proposition (so “message” is institutionally a homonymy bearer). I keep this double acceptation not because a homeopathic war against homonymy suddenly seduced me but because there are peculiar problems (for instance the problem concerning the objects of truth) whose discussion is highly assisted by some homonymy bearers; in fact only if the problem can be formulated by an ambiguous word, its solution is not anticipated by the same formulation.

1.9.2. In natural languages homonymy bearers are very numerous. So then at first sight, it may seem a miracle that in our current linguistic practice we can deal with them without being continuously misled towards wrong interpretations. No miracle, since we are helped by the rule usually called “principle of charity”, according to which any sentence must be interpreted in the way optimizing the adduced meaning. I prefer to call “criterion of (interpretative) collaboration” such a rule and, above all, I wish to extrapolate its range to non-linguistic contributions too, since actually our interpretations are influenced by non-linguistic sources of information too. For instance

(1.iv) I wish to buy a black pen

is spontaneously (and correctly) interpreted in two different ways if (1.iv) is uttered in a swan-breeding or in a writing material shop. Analogously, since neither a female swan nor an instrument for ink writing can reasonably enclose a pig, the reading of

Bob wishes to buy a pen to enclose his pig

suggests the perplexed interpreter to ascertain whether “pen” adduces a third meaning; and actually any perplexity disappears as soon as he learns that a pen is also a small enclosure for animals.

1.9.2.1. An even more evident application of (1.iv) concerns situations where the homonymy bearer has different syntactic statuses. For instance, both

(1.v) Bob felt cold

and

(1.vi) Bob’s felt has been manufactured in France

would be word salads if we should read “felt” as a substantive in (1.v) or as a voice of “to feel” in (1.vi). But by the simple fact that the two reciprocal readings lead to sensible statements, such readings are automatically assumed. Anyway the criterion of collaboration is peculiarly applied to homonymy bearers with a common syntactic status, that is situations where the possible interpretations concern an anyhow well formed sentence.

The above decision (§1.6.1) to specify the language of reference only where necessary can be legitimated just by an appeal to the criterion of collaboration; since normally if we read in $L_1$ a proper $L_2$-sentence, we face a senseless expression, it is our duty to choose the language the sentence under scrutiny belongs to.

1.10. Under the current notations italics, though implicitly, is a polyvalent type. In fact, besides some fanciful applications I neglect, it is used

- for symbolic expressions
- for expressions to emphasize
- for expressions belonging to foreign languages
- for direct quotations

(in the last chapter we shall see that italics is also the type Goedel chose for arithmetization).

I disapprove of this multipurpose practice which, among other inconveniences, forbids focusing more subtle distinctions (for instance, direct quotations are a wide topic where heteroquotations ought to be distinguished from autoquotations, sub-quotations and so on). Nevertheless I will resign to follow that practice with the only exception of the bold type for expressions to emphasize. My resignation follows from the conviction that to introduce a really satisfying set of graphical conventions would mean to lose even the best disposed of my few readers. The aim of these simple considerations, then, is to remark that my decision is a bitter compromise.

1.11. On the contrary I cannot accept the current diacritical notations, since they are quite insufficient for my requirements. To enrich them is an indispensable step towards refining any approach to logic. I realize that my claim is unusual and perhaps presumptuous (ideas are lacking, not the means to express them, annotated Leonardo); nevertheless I hope that the next pages will legitimate it. Indeed mankind did without zips when only buttons existed, and did without buttons too before their invention, yet even emperors got dressed; nevertheless so unquestionable an evidence neither entails that buttons and zips are useless, nor that a note like dresses are lacking, not the means to close them eliminates the problem.

More than seventy years ago Morris, genially, denounced the poorness of the means by which natural languages can speak of themselves: but till now what improvements have been realized? And almost half a millennium ago Montaigne, genially, wrote: "la plupart des troubles du mond sont grammariennes" (the troubles of our world are for the most part grammatical). With the aggravating circumstance, nowadays, that symbolic troubles add themselves to
1.11.1. Also the worst symbolic convention, that is the universal habit according to which affirmation is expressed by omitting the symbol of negation, can be classified as a failure in the diacritical notations.

1.12. Orthodoxy states that the first aim of a formalized language is to settle a clear pattern able to explain how natural languages work and to show their limits. Although in due course formalized languages will be used massively, I exhort the reader not to mythicize them: just as a definition cannot allow new theorems to be proven, a formalized language cannot open new heuristic horizons. In fact, if natural languages should be conditioned by some limits intrinsically permeating our mind, the same limits would condition too whatever artificial language we can elaborate; and if such limits can be by-passed, then we have the faculty of refining directly natural languages.

Anyhow the final chapters will show that the main danger entailed by formalized languages is their absolute (yet only presumed) trustworthiness.

1.12.1. A natural language can be compared to a house obtained by successive enlargements and repairs from a very old nucleus: it is not rationally disposed, is lacking of some comforts et cetera, nevertheless the centuries of its story gave it an already tested usableness. A formalized language, on the contrary, can be compared to an aseptic scale model very fit for certain applications, but surely unfit for lodging evacuees.

1.12.2. Incidentally, The current claim that our thinking activity is necessarily linguistic seems to me radically untenable: if it were, by what kind of activity should natural languages be born?
CHAPTER 2
DIACRITIC SYMBOLOGY

Philip of Macedonia, warned by the oracle to beware of quadrigae, ordered their destruction and even avoided the so-called region, nevertheless he was stabbed by Pausania’s dagger, whose hilt represented a quadriga.

Object, name, representation.

2.1. Diacritical and syncathegorematic symbology is a rather neglected matter although its shortcomings are much more detrimental than the shortcomings of the terminological endowment. For instance, it seems to me like a grotesquery that the English lexicon is so rich in terms expressing chromatic properties or fighting actions and so poor in terms expressing propositional connectives (disjunctions, in particular). Maliciously, I might suppose that the more ambiguous is a symbology, the more easy is to hide behind its ambiguity an insufficient perspicuity.

Mindful of the mentioned Morris’s complaint I start improving the ‘metalinguistic’ symbology.

2.1.1. The meticulous analysis of the distinction between tokens and types (Peirce), that is the distinction between sign event and sign design (Carnap) is deferred to the next chapter. For the present I speak of a word to mean just the type, that is the abstract syntactical entity whose instances are its various tokens.

2.2. In §2.1 above I enclosed (2.i) metalinguistic within single inverted commas to mean that (2.ii) metalanguage is a rough notion. Awaiting for a better convention I assume dialanguage as a synonym of (2.ii) in such a rough acceptation, and hyphens as a dialinguistic operator. I make my point clearer.

2.2.1. The scholastic distinction between suppositio materialis and suppositio formalis is the distinction between use and mention. For instance (2.iii) indulgent is used in (2.iv) Bob is indulgent since (2.iv) speaks of Bob’s indulgence; on the contrary, both in (2.v) -indulgent- is trisyllabic and in (2.vi) -indulgent- is commendatory. According to the current symbology, hyphens could be substituted by quotation marks (“…”), both in (2.v) and in (2.vi); but I claim that this procedure is deeply misleading, since what (2.v) speaks of (a word, a syntactic entity) is not what (2.vi) speaks of (a meaning, a semantic entity). In fact, just as (2.vii) “indulgent” is trisyllabic is an unexceptionable statement, and the meaning of “indulgent” is trisyllabic is a senseless one (better: a sortally improper one) because meanings are not syllabic entities, the meaning of “indulgent” is commendatory is an unexceptionable statement, and (2.viii) “indulgent” is commendatory is a sortally improper one. In order to prove this last issue, it is sufficient to replace (2.iii) with (2.ix) fond and to remember the archaic meaning (-silly-) of (2.ix); since this meaning is not at all commendatory, if we insist claiming that the predicate refers to the word, we cannot avoid deriving “fond” is commendatory and not commendatory so falling into incoherence. On the contrary, as soon as we acknowledge that commendatory refers to meanings and that (2.ix) adds two different meanings, the present meaning of “fond” is commendatory and the archaic meaning of “fond” is not commendatory become two sortally proper (and even true) statements.
2.2.2. To realize that we mention a linguistic entity with reference sometimes to the word and sometimes to its meaning (that is to the piece of information adduced by the same word) is so fundamental an achievement that I try to impress it through another example.

The editor is criticizing the too inflamed tones of an article:

- pusillanimous- is offensive

he says, suggesting the choice of a more moderate adjective. Yet the editor is referring to a meaning; in fact to substitute “pusillanimous” with “coward” would be derisive, since the change of the word would not entail any change in the meaning.

2.3. On this ground I say that a statement (is protolinguistic iff it does not concern a linguistic referent (giving “linguistic” its widest acceptation)

- is dialinguistic iff it does concern a linguistic referent.

Then, for instance, while (2.iv) is a protolinguistic statement, (2.v) and (2.vi) are dialinguistic ones.

Moreover I say that a dialinguistic statement

- is metalinguistic iff, just as (2.v), it concerns a syntactic referent (an expression)

- is hyperlinguistic iff, just as (2.vi), it concerns a semantic referent (a piece of information, a meaning).

Analogously I speak of the protolinguistic (or dialinguistic, metalinguistic, hyperlinguistic) faculties of a language.

2.4. I call “asterisks” the (mute) symbol 

* ... *

and I agree that asterisks are an abbreviation of

(2.x) the meaning of “ ... ”

(that is: I introduce asterisks as a semantic symbol, just as quotation marks are a syntactic symbol and hyphens are a bivalent, therefore an ambiguous symbol). Consequently, for instance

(2.xi) *indulgent* is commendatory

is the accurate symbolization of (2.vi).

There are two alternatives to the introduction of a specific (and mute) semantic symbol, and precisely

- to continue accepting quotation marks as an ambiguous symbol whose interpretation is sometimes syntactic (to mean the enclosed expression) and sometimes semantic (to mean the piece of information adduced by the enclosed expression)

- to use an explicit formulation like (2.x).

Yet both of them are unsatisfactory. The former because it does not overcome a general ambiguity, therefore it does not allow the great theoretical advantages we can draw from an accurate symbology. The latter because it does not account for the evidence that sentences like (2.v), where no “the meaning of” occurs, are perfectly grammatical in our usual practice. In other words: to introduce a periphrasis would not account for the silent ambiguity hidden under the lacking distinction between a metalinguistic and a hyperlinguistic mention. Only the introduction of a mute symbol different from the metalinguistic one can overcome the ambiguity affecting current languages; in this sense while (2.viii) is a wrong translation of (2.vi), (2.xi) is the right one.

2.4.1. The tripartition sign-meaning-referent (starting from the lektion of the Stoicism until the triangle of Ogden and Richard) is well known. Asterisks fulfil a symbolic lack, so satisfying the mentioned Morris’s remark and, above all, so contributing to a clarification of our ideas about a very insidious topic.

2.4.2. An easy analogy. Besides using (spending) a banknote, people can speak of it. But while a discourse between two mint technicians appreciating its filigree is ‘syntactic’, a discourse between two housewives complaining its poor purchasing power is ‘semantic’.

2.4.3. Here I could exhibit a lot of illustrious quotations where the lack of a semantic symbol constraints the authors to untenable formulations. I avoid it mainly because I am interested in exposing my ideas, not in censuring methodically other people’s ones. Such a censure will be carried out only when necessary to support mine, and anyhow without any gossipy pleasure (in the most tedious way, then). Yet, if I were quite frank, I ought to confess that the strongest reason is the hope that my liberal attitude be reciprocated.

2.5. All the dialinguistic symbols (hyphens, quotation marks and asterisks) have a substantivizing effect. For instance ($§2.7 as for the use of new lines)

\textit{indulgent}

is an adjective, but

\textit{-indulgent-} “indulgent” *indulgent*

are substantives; the same syntactic well formation of (2.v), (2.vii) and (2.xi) legitimates this conclusion. The obvious reason is that if we speak of words or of meanings, we speak of objects, therefore we need nouns.
2.5.1. The distinction between metalanguages and hyperlanguages, of course, is not the well known distinction between different metalinguistic orders. For instance "indulgent" is an adjective is a true metalinguistic statement and ""indulgent"" is a substantive is a true meta-metalinguistic statement, but surely it is not a hyperlinguistic statement, since it does not concern semantics.

2.5.2. Dialinguistic symbols are freely concatenable. For instance "*nor*" is a substantive is a formally correct (and true) statement saying that the expression obtained by enclosing between asterisks nor is a substantive naming the meaning of that conjunction. More about this point in §3.4.

2.5.3. There are contexts where the clear distinction between quotation marks and asterisks is nearly impossible, just owing to the ambiguity of the message.. For instance, strictly, (2.xii) hyper derives from υπερ is a syntactically incorrect sentence (it is a word salad) since the first and the last words are prefixes, not substantives; and the mere criterion of interpretative collaboration makes (2.xii) understandable. Yet both "hyper" derives from "υπερ" and *hyper* derives from *υπερ* are tenable correct formulations; while the former speaks of a historical link between signs, the latter speaks of a historical link between notions. The interesting peculiarity of (2.xii) is that the lack of a dialinguistic symbol, superficially, seems to concern only the first word, because the typographical peculiarities of the last word (the Greek type) favours (even improperly) its objectification. In other words, though "hyper" derives from υπερ instances a syntactically incorrect expression too, such a kind of incorrectness is rather current.

2.6. Cartesius taught us that the existence of our mental activity is even more unquestionable than the existence of the world around us. And a mental activity is the most essential factor in any linguistic process. In this sense to neglect such a factor entails the ‘topological deformation’ (from tridimensionality to bidimensionality) already denounced (§1.1). The worrying meta consideration is that the tacit acceptation of the inadequateness affecting the current (natural and artificial) languages reveals the intrinsic poorness of the ideas leading the approach. A poorness particularly prejudicial for logic, because to reason on the logos is necessarily to reason dialinguistically. First of all, in order to bridge the gap we must recognize that, besides the suppositio materialis and the suppositio formalis, a suppositio informationalis is also necessary (I apologize for the disconcerting expression, but more or less all new expressions, at their start, are disconcerting). Asterisks are just the first step to reaching this goal.

2.7. A very precious dialinguistic operator (whose detailed analysis, as far as I know, has never been proposed) is represented by the new line. In the current linguistic practice the new line is a highly polyvalent operator; the paradigm of its various functions follows.

2.7.1. The metalinguistic use of a new line is exemplified by (2.i), since what can be enclosed within single inverted commas is an expression, not a meaning. A metalinguistic new line, of course, is an alternative to quotation marks; and actually (though at first sight more equivocal) (2.xiii) In § 2.1 I enclosed “metalinguistic” within single inverted commas is a perfectly equivalent formulation (a pedantry: the same (2.xiii) is a new example of a metalinguistic new line, since it is used to speak of a formulation).

The comparison between (2.i) and (2.xiii) emphasizes that new lines, just because they consist of a spatial disposition without any intervention of graphic elements, are the cleanest dialinguistic operator.

2.7.2. The hyperlinguistic use of a new line is exemplified by (2.ii) since what can be a rough notion is a meaning, not a sign. A hyperlinguistic new line, of course, is an alternative to asterisks; and actually (2.xiv) *metalanguage* is a rough notion is an alternative formulation (a pedantry: (2.xiv) too is an example of a metalinguistic new line, since it too speaks of a formulation).

2.7.3. The ambiguous dialinguistic use of a new line can be instanced by writing that
Bob bought a pen is the message under scrutiny; in fact the same ambiguity of *message* forbids a specifically metalinguistic or hyperlinguistic interpretation of the new line. In this ambiguous use, the new line can be replaced by hyphens.

2.7.4. The mere emphasizing (therefore protolinguistic) use of a new line can be exemplified by enclosing (2.i) between quotation marks or (2.ii) between asterisks; in these new formulations the new line becomes only an emphasizing operator devoid of any dialoguistic import.

2.7.5. A new line is sometimes recycled, in the sense that a further reference may change the original use. A clear example is in Chapter 1, where (1.i) is introduced as a metalinguistic new line (if the speaker utters (1.i) ...) but few lines below is recycled hyperlinguistically (the piece of information... (1.i) ...). Did the reader realize it?

2.7.6. Of course, owing to the proliferation of their different uses, new lines, so to say, are strong homonymy bearers. Yet a homonymy bearer can act as a catalyst of incoherence only on condition that we are unaware of its status. And the proposed analysis not only allows us to exclude any risk of unawareness; it stimulates too a critical interpretation of the various new lines occurring in the text (for instance the new lines of §2.5, §2.5.1, §2.5.2). Anyway, in order to help the reader, from the start I will comment on the more interesting applications.

2.8. A meaning postulate (Carnap) associates a certain piece of information to a certain word, and in §1.6.1 I called “semantic” (symbolically “σ”) the respective relation. Obviously different languages (different linguistic codes) may associate different meanings to a same word. For instance “largo” adduces *long* in Spanish (S) and *broad* in Italian (I). Therefore

(2.xv) *largo* (metalinguistic new line) is an elliptic expression since the omission of the linguistic code does not allow to single out the meaning (2.xv) speaks of. In order to overcome this ambiguity I agree that asterisks must be completed by an index specifying the linguistic code of reference; accordingly

* lasso*<sub>S</sub> = *lungo*<sub>I</sub> = *long*<sub>E</sub>

is a sentence where the agreement is applied. Another way to present the same passage is referring the index to the σ-relations, so writing

σ<sub>S</sub>(“largo”) = σ<sub>I</sub>(“lungo”) = σ<sub>E</sub>(“long”)

in order to mean that the informational nucleus associated in Spanish with “largo” (merely emphasizing new line which becomes a metalinguistic new line if we cancel the pair of quotation marks) is the same associated in Italian with “lungo” et cetera. In other words: the English code book assigns to “long” the same attribute (characteristic, quality) the Spanish code book assigns to “largo” et cetera. For the sake of concision the index may be omitted when the linguistic code of reference is the contextual one (English, in this case).

2.8.1. Here too homonymy is a disagreeable enemy; in fact, for instance, in spite of its index,

(2.xvi) *sole*<sub>E</sub>

is an ambiguous expression, since in English “sole” is a homonymy bearer.

A first way out from this impasse is the appeal to another language where the various meanings are adduced by different words, as, with reference to (2.xvi),

* solo*<sub>I</sub>

(alone)

* suola*<sub>I</sub>

(bottom of a shoe) and

* sogliola*<sub>I</sub>

(flatfish).

Reciprocally as for

(2.xvii) *lira*<sub>E</sub>

(*lyre* and *lira*).

A second way out is the enrichment of the homonymy bearer by different graphic elements. So, for instance, the ambiguity of (2.xvii) disappears as soon as a “lira” (that of course, owing to the index, would name my troubled national (ex)currency) were distinguished from a “lira” (that would name the musical instrument played by Nero). Anyhow, since these minutiae do not influence my discourse, I remarked them only to underline (meta-theoretically) that also expressions like

the meaning of “...” in L

or like

the piece of information adduced by “...” in L
are ambiguous when the expression within inverted commas is a homonymy bearer in \( L \).

2.9. Owing to the crucial role of asterisks I pay attention to the following objection.

There is no necessity to introduce a semantic symbol because even if actually some attributes pertain to meanings, no one can forbid us from defining their correspondents pertaining to the respective words: For instance, we can define

\[(2.xviii) \text{ commendophorous} \]

(ambiguous new line, since we have not yet specified whether words or meanings are the real objects of definitions) by agreeing that an adjective is commendophorous iff its meaning (iff the attribute it adduces) is commendatory. Then the message transmitted by \((2.xi)\) is also transmitted by

\[(2.xix) \text{ “indulgent” is commendophorous} \]

and \((2.xix)\) allows us to reason unobjectionably on signs, thus rendering asterisks a superfluous device.

As we shall see, the objection is very momentous; yet my first reply is playful. Ava, Bob’s wife, is suspected of conjugal infidelity by her father-in-law Anyl (\textit{omen omen}). Indeed he thinks that a true gentleman cannot pay a detective to shadow a lady; then, since notoriously if a wife is conjugally infidel her husband is a cuckold, Anyl commissions the detective to shadow Bob in order to ascertain his eventual cuckoldness.

What of Anyl’s idea?

2.9.1. Awaiting for a stricter approach to the notion of a dilemma (§6.7), here I recall Frege’s notion of a \textit{Satzfrage}: \textit{a Satzfrage contains a demand that we should either acknowledge the truth or reject it as false} [Black and Geach, 1960]. Roughly, a dilemma is the cognitional situation where two opposite alternatives (the horns of the dilemma) are considered, but none of them is asserted. So, for instance

\[(2.xx) \text{ is Ava an infidel wife?} \]

is Anyl’s dilemma (the interrogation point is the spontaneous symbol to express a dilemma).

The (right or wrong) solution of a dilemma is the assignation of a ‘truth-value’ to its horns.

The (basic) datum of a dilemma is the piece of information on whose ground the solution is attained.

The core of a dilemma is the fact whose knowledge constitutes the (basic) datum.

Therefore, while the core of Anyl’s dilemma is Ava’s behaviour, the datum is the piece of information on whose ground one horn of the dilemma is (rightly or wrongly) asserted. Evidently *core* and *datum* are two different notions; an easy way to understand the difference is to think of an untrustworthy detective who, instead of pursuing the datum of Anyl’s dilemma, would pursue its core on his own account.

2.9.2. If the reality were made of monads, a dilemma could be proposed only by focusing on the concerned monad. Reality is an interrelated network of nearly unconceivable fineness, and just owing to these interrelations we can formulate various dilemmas whose core (then whose datum) is anyhow the same (I recall the Aristotelian \textit{pollakos legomenon}, that is what can be said in many ways). In this sense to define consequent properties pertaining to Ava’s husband (is he a cuckold?) or to her sons (are they sons of a promiscuous woman?) and so on, is a procedure which modifies only the superficial aspect of the discourse, since they all depend on the same factual evidence (they all concern Ava’s corporeal liberality, which is anyhow the common core of all dilemmas).

The theoretically momentous distinction between pertinence and regard (of an attribute) is based just on this discriminating element. So, for instance, we say that cuckoldness pertains to husbands, but regards wifes. Consequently, among the various dilemmas concerning the same core, we can privilege the one where pertinence and regard coincide, that is, in the contingency, \((2.xx)\).

Coming back to the objection of §2.9, \((2.xviii)\) legitimately defines an attribute pertaining to adjectives, yet no definition can modify the core of the respective dilemma, whose solution continues depending on the commendableness of the attribute adduced by the adjective whose ‘commendophorablesness’ we have to ascertain. Then , on the basis of the considerations proposed in §2.4, the use of a mute semantic symbol is not by-passable.

2.10. Synonymy is another topic directly affected by the mentioned distinction. In fact, as soon as we realize that any formal transformation of signs presupposes implicitly the signic nature of the same signs (no thing without a semantic dimension can be a sign) a conclusion becomes evident: synonymy is a relation whose pertinence is syntactic, but whose regard is semantic. I make my point clearer.

The formal approach to signs leaves their meanings out of consideration; nevertheless it concerns \textit{signs}, which is to say objects having that certain characteristics. We can reason on European citizens leaving their nationality out of consideration (for instance stating that they have the right to cross freely the European frontiers), yet we cannot extrapolate automatically our statements to persons who are not European citizens, so then we can leave their nationality out of consideration \textbf{provided that} they have a certain requisite. Analogously we can leave the meanings out of consideration, provided that the objects we are reasoning about have a meaning.

In formal treatments definitions are introduced as abbreviations, and abbreviations are managed through a purely syntactic approach. Yet no syntactical approach can overcome the semantic role of a given sign: to introduce an abbreviation in a formal theory is to agree that in every interpretation of the same theory, definiendum and definiens adduce the same piece of information, though any specific piece of information is left out of consideration.
In other words, the way we deal formally with signs (leaving their meanings out of consideration) is far from being the way we deal with arabesques (where the same notion of an abbreviation is meaningless).

2.10.1. The same current symbolization of synonymy through something like 
\[ x \equiv y \]
evidences my claim. Such a symbolization is surely misleading since manifestly the two synonymous expressions are not identical. As what is identical is their meaning,
\[ *x* \equiv *y* \]
is the correct symbolization; therefore by 
\[ x \equiv y \text{ iff } *x* \equiv *y* \]
I introduce the symbol for synonymy
(2.xxi)
which is called “inverted arrows”. By definition (2.xxi) is a symbol of equivalence having syntactic pertinence (it is established between signs) and semantic regard (it depends on a relation between meanings). It plays an important role in definitions (§4.6) allowing to adequate symbolic formulations to usual expressions.

2.10.2. An etymologic pedantry. Just as *synonymous* pertains to signs (which are synonymous iff they adduce the same meaning), *homonymous* pertains to meanings (which are homonymous iff they are adduced by the same sign). Then, since the two attributes pertain to heterogeneous referents, to give them a common ending (“onymous” from “ονοµα”, where *ονοµα* Gr = *noun*) is misleading; while it is etymologically correct to say that different meanings adduced by a same word are homonymous, it would be better to say that different words having the same meaning are homosignificant. But of course this is not a proposal, this is only a pedantry, I repeat.

2.10.3. Synonymy (homosignificance) and homonymy are not absolute relations; they must be referred to a language (*sideboard* and *belief* are homonymous in Italian, where “credenza” adduces both of them, but evidently they are not homonymous in English). And just because we write signs, not meanings, synonymy can be directly formulated (“wide” and “broad” are synonymous), but homonymy cannot (*pen* and *pen* are homonymous?). For instance,

in English *pen* is both *female swan* and *instrument for ink writing*

or

“pen” is in English a homonymy bearer for *female swan* and *instrument for ink writing*

are two correct (but indirect) ways to formulate a homonymy.
CHAPTER 3
DIALINGUISTIC REFINEMENTS

3.1. This chapter is mainly devoted to showing how poor is the current dialinguistic symbologic endowment and how fuzzy are the ideas of celebrated authors on the matter.

A language, obviously, can concern several different sensorial channels (for instance a Braille text is read through touch), yet the following analysis deals only with written and spoken languages.

I start from the well known distinction between tokens (sign events) and types (sign designs). both graphic and phonic token are considered as utterances. I hope the reader will forgive the frivolousness of the following examples (que le lecteur ne se scandalise pas de cette frivolité dans le grave, Baudelaire would comment), since its aim is just to involve our usual lingustic behaviours.

3.2. Hyphens have been introduced as a diacritic symbol that generically stands for quotation marks (when the referent is the enclosed term) or for asterisks (when the referent is the meaning adduced by the enclosed term). Yet a much more detailed analysis is possible.

3.2.1. The press-corrector who says
\[\text{-shuFFLer-} \text{ is misprinted}\]
refers to (and exactly to) the graphic token appearing in the single copy typewritten he is reading. Therefore if we should agree a specific diacritical symbol (quadruple inverted commas, say) to mean the mentioned word where it occurs as a graphic token, the hyphens of (3.1) ought to be substituted just by quadruple inverted commas.

3.2.2. The lady I am helping in the solution of a cross-word puzzle tells me
\[\text{-shuffler-} \text{ is too short}\]
explaining that the letters are eight, while the cases to fill are nine.

But what exactly is she referring to? Surely she does not refer to something phonic, because if the sound of “sh” were written in English as it is in German (“sch”), the length of the suggested word would be exact. Reciprocally, any modification of English orthoepy (for instance to pronounce “shuffler” like we now pronounce “skiu ffler”) would be of no moment as for the shortness of the suggested word. She is referring to something graphic; but if the referent were a token, then the attempt at filling the nine cases with another token would not be an absurdity. She (perhaps without realizing it) is exactly speaking of the graphic type, so that, if we should agree a specific diacritical symbol (triple inverted commas, say) to mean the mentioned word where is occurs as a graphic type, the hyphens of (3.ii) ought to be substituted by triple inverted commas.

3.2.3. The director who reproaches the bad actor with
\[\text{-shuffler-} \text{ thus you have to pronounce, with only a nuance of disdain}\]
refers evidently to a phonic token, that is to the sound event consisting in the same utterance of the master. Therefore if we should agree a specific diacritical symbol to mean the mentioned word where it occurs as a phonic token, the hyphens of (3.iii) ought to be et cetera.

3.2.4. The bad actor who, after hundred fruitless rehearsals, dares to say
\[\text{-shuffler-} \text{ is cacophonous, -trickster-} \text{ is better}\]
is speaking of the phonic type. Of course also the hundred phonic tokens he uttered were cacophonous, but this is only a consequence of the phonic type cacophony, and the best proof is simply that, because the cacophony of a type entails the cacophony of all its tokens, his intuition disclosed him the absolute uselessness of new attempts. Therefore if we should agree et cetera, the hyphens of (3.iv) ought to be et cetera.

3.2.5. The scholar who says
\[\text{-shuffler-} \text{ is a noun}\]
refers to the type in its generic acceptance. Here too, of course, if a term is a noun, also the respective graphic (or phonic) type is a noun, yet to read (3.v) with reference to a specifically graphic (or phonic) dimensions would mean to add an abusive specification, then to adulterate the original message. Therefore in (3.v) hyphens are properly used.

In §2.5.3 another context legitimating the use of hyphens is instanced. Analogously -orthodoxy- derives from -ορθοσ- and -δοξα-, may be interpreted syntactically (the mentioned English word, quite independently of any semantic consideration, follows from the mentioned Greek words) as well as semantically (*orthodoxy* follows from *ορθοσ* and *δοξα*, that is from *right* and *opinion*)


3.2.6. The director who, offended by (3.iv), thunders out (3.vi) -trickster- is too offensive is speaking of a meaning (§2.2.1) therefore asterisks are the right specific diacritical symbol to replace the hyphens in (3.vi).

3.3. Of course I have no intention of agreeing and dragging on a so Pharaonic diacritical symbology. I maintain the only distinction between quotation marks and asterisks since it is actually indispensable to develop my theses (a simple analogy which can enlighten this indispensability will be proposed in §3.4). The aim of the more detailed paradigm above was to show not only that ordinary semantics is rough, but also that this roughness is usually unperceived and then that the approximations I often accept are not born by a congenital incapacity to refine the analysis.

3.3.1. I am not a linguist (and some rude person might even think that even as a logician ...). My interest in linguistics follows not only from the (internal?) fact that a natural language is my main instrument to express rather difficult theses, but also from the (external?) fact that linguistics (through semiology) belongs to every logic which refuses to be reduced to pure formalism. Therefore, just as it is highly restrictive studying linguistics without involving semiology and studying semiology without involving logic, it is highly restrictive studying logic without involving gnosiology, and studying gnosiology without involving ontology (I do not dare to follow with a last step from ontology to metaphysics in order to preserve my last reader). The viewpoint of linguistics is undoubtedly able to focus on fine miniatures, yet it grows dim where the problems concern wider horizons (even the great Saussure speak of the semantic mystery, if I remember correctly): a microscope is not the best instrument to observe elephants fighting. To look at a territory through the mentality of a resident who never left it or through the mentality of a far-coming and far-going traveller entails momentous differences. A nearly sacred example: until today the common mistake affecting all the proofs I know of Goedel’s Incompleteness Theorem succeeded in hiding itself mainly because the same proofs have been scrutinized by mathematically sharp but dialinguistically somewhat dull eyes. Even the image reflected by a deforming mirror is perfect for those who are ascer taining whether it respects exactly the laws of optics.

3.4. Here is the analogy. We took a magnetic picture \( p_1 \) of a certain object \( c \) (the Coliseum, say). Through a fit apparatus able to read the magnetic micro-arabesque on the tape we project an illuminated \( c \)-image on a screen. The analogy associates the magnetic micro-arabesque with the name of the object (that is “\( c \)”), and its image on the screen with the piece of information adduced by “\( c \)” (that is “\( c^* \)”). In other words, the analogy associates the apparatus to our semantic faculty.

During the projection of \( p_1 \), we take a picture \( p_2 \) of the whole scene, so that when \( p_2 \) is projected, we see both the image of the \( p_2 \)-micro-arabesque (that is * “\( c \)” *) and the image of the image of Coliseum (that is * “\( c^* \)” *). Two words to avoid any confusion between * * “\( c^* \)” * and “\( c^* \)” : if, after \( p_1 \), the Coliseum were destroyed, we could not take new pictures of the intact monument, but we could continue to take new pictures of its intact image on the screen during the projections of \( p_1 \), therefore what we see on the screen during the projection of \( p_2 \) is not the image of the Coliseum (that is “\( c^* \)”), but the image of its \( p_2 \)-image (that is * * “\( c^* \)” *).

Analogously “ \( “ \) “ \( c \)” and “ \( “ \) “ \( c^* \)” ” are associated respectively to the magnetic \( p_2 \)-micro-arabesque impressed by the \( p_2 \)-micro-arabesque corresponding to “\( c \)” , and to the magnetic \( p_2 \)-micro-arabesque impressed by the \( p_2 \)-micro-arabesque corresponding to “\( c^* \)” ,

I do not insist on the sequence because the exponential function increases swiftly (although less swiftly than Cantor’s opinion). Anyhow the crucial passage is clear: a dialinguistic symbology formed by only one symbol would be highly inadequate.

3.5. In order to refine the distinction among various dialinguistic orders, I start from a quotation: (3.vii) If we describe in English the grammatical structure of modern German ... then German is our object language and English is our metalanguage (Carnap, 1942, §1).

According with (3.vii) der Himmel ist blau is a protolinguistic sentence of the object language the sky is blue is a protolinguistic sentence of the metalanguage, “blau” ist ein Adjectiv is a metalinguistic sentence of the object language (which just possesses metalinguistic faculties), “blue” is an adjective is a metalinguistic sentence of the metalanguage which speaks of itself as object language, and (3.viii) “blau” is a German adjective is a metalinguistic sentence of the metalanguage which speaks of the object language. Finally
“blau” means in German what “blue” means in English
that is, in short,
“blau” means *blue* in German
is a hyperlinguistic sentence of the metalanguage (which just possesses hyperlinguistic faculties) concerning a German semantic relation (that is a $\sigma_D$-relation whose correlatum is singled out through an English connotation).

3.5.1. The intervention of a third language does not modify the structure of the discourse. For instance
““blau” is a German adjective” è un enunciato inglese
is a meta-metalinguistic sentence where English (metalanguage of German) is the object language of Italian.

3.6. Now let me agree that
lulù
is the name in English of the German adjective “blau”. Then
(3.ix) lulù is a German adjective
becomes a perfect synonym of (3.viii). The discrepancy between (3.viii) and (3.ix) depends on the discrepant criterion
adopted in order to form the metalinguistic lexicon through which to speak of the object language one. While I call
“standard” the criterion of (3.viii), that obtains any metalinguistic word by enclosing the object word within quotation marks, I call “autonomous” the criterion of (3.ix), that coins a specific metalinguistic word for any object word.
Of course the autonomous criterion is too expensive to result practicable: yet to not have spoken of it would have been a censurable theoretical omission.

3.6.1. A pedantry. The autonomous criterion would also forbid the use of asterisks. In fact while
*blau*$_D$ is a chromatic notion
is a correct (a proper) sentence,
*lulù* is a chromatic notion
is an incorrect (an improper) one; in fact to be correct “lulu”, instead of being the name (in English) of a (German) adjective aducing a chromatic notion, ought to be an adjective aducing a chromatic notion. The name relation is the most insidious of logic because mistaking two referents separated by a dialinguistic order is the paramount trap.

3.7. Let
(3.x) $\Gamma(b)$
be a well formed formula (wff) belonging to a formal language $L$. Then
(3.xi) the expression formed by concatenating horizontally from left to right the third capital letter of the Greek alphabet and the second small letter of the Latin alphabet enclosed within parentheses
describes (3.x) in English. And as soon as we agree upon a symbol for the concatenation ("\^", say)
(3.xii) "$\Gamma ^ {\wedge} " ^ {\wedge} ^ {\wedge} ^ {\wedge} ^ {\wedge} " "$ ^ {\wedge} " "$ ^ {\wedge} ^ {\wedge} " "$ ^ {\wedge} ^ {\wedge} " $"
becomes the symbolic translation of (3.xi) in $ML$. In its turn
(3.xiii) " " $\Gamma ^ {\wedge} " ^ {\wedge} ^ {\wedge} ^ {\wedge} ^ {\wedge} " "$ ^ {\wedge} " "$ ^ {\wedge} ^ {\wedge} " "$ ^ {\wedge} ^ {\wedge} " $"
becomes the $MML$ expression describing symbolically (3.xii). And so on.
Yet if I were even more meticulous, I should remark that, strictly, another convention is tacitly understood in order to make (3.xii) the description of (3.x) and (3.xiii) the description of (3.xii): in fact it would be sufficient to agree that while $L$ must be read from left to right, $ML$ must be read from right to left, to realize that
$\triangledown (\Gamma)$
is the $L$-expression described by (3.xii) et cetera.

3.8. Now I quote two celebrated authors to show their dialinguistic fuzziness (both quotations are re-translations from the Italian translations).

3.8.1. Mendelson (1964 § 1-4), writes
Now we introduce by definition, other connectives
D1 $\ (A\&B)$ for $\sim(A\supset\sim B)$
...........................
the meaning of D1 is: for every wff A and B, “(A& B)” is an abbreviation for "$\sim(A\supset\sim B)$". and in a footnote

When we say that “(A&B)” is an abbreviation for "$\sim(A\supset\sim B)$”, we mean that “(A&B)”
must be assumed as another name ... for the term "$\sim(A\supset\sim B)$".... These conventions are quite natural and would have not been noted by the majority of readers if they were not explicitly remarked. Anyhow further explications can be found in Carnap ...
The quotation is interesting because Mendelson falls exactly into the trap he is warning the reader not to fall into. In fact the last occurrence of quotation marks is wrong: expressions like

(3.xiv) when we say that X is an abbreviation for Y we mean

that X must be assumed as another name for Y

are intrinsically affected by a mistake between *to abbreviate* and *to name*. While an abbreviation belongs to the same dialinguistic order (to the same language) of the expression it abbreviates, a name belongs to the successive order (to its metalanguage). Let me recall (3.xiv) when we say that “JFK” is an abbreviation for “John Fitzgerald Kennedy” what do we mean? That “JFK” must be assumed as another name for “John Fitzgerald Kennedy” or for John Fitzgerald Kennedy?

This is an unobjectionable point which is perfectly focused just by Carnap (1937 §42), whose metalinguistic perspective, in my opinion, is usually punctual and trustworthy. Of course his perspective too is limited by the lack of a semantic symbol, but this limitation is not to be confused with a bad use of the syntactic one.

3.8.2. In his turn Shoenfield [1967] writes

We say \textit{unary} for 1(ary and \textit{binary} for 2-ary

(§ 2.1),

Now we introduce $\rightarrow$ whose meaning is \textit{if ... then}

(§2.2),

Now we define the \textit{recursive} functions

(§ 6.2). Since, respectively,

We say “unary” for “1-ary” and “binary” for “2.ary” ...

Now we introduce “$\rightarrow$” whose meaning is *if ... then*

Now we define the \textit{recursive} functions

are correct re-formulations of Shoenfield’s text, we deduce that italics is used as a metalinguistic, as a hyperlinguistic and as a merely emphasizing operator. My new lines too present this polyvalence; the difference is that Shoenfield does not spend even a word explaining his convention, and that, anyhow, it is not well applied (why are “l(ary” and “2(ary” not in italics?).

Furthermore (and mainly) another convention is quite unacceptable. I allude to his assumption according to which the formal expressions are also names for themselves. This means falling into the worst kind of homonymy, that is the autonomic one. Indeed he claims that no homonymy arises, since the context allows us to avoid any ambiguity: in fact a formal expression must be interpreted autonymically (as a name for itself) only where it occurs in a non-formal context. Which this is not. For instance (his §2.1)

(3.xv) \textit{... a representation ...} is an assignation \textit{...} If $F$ designates a representation and $F$ assigns ...

contradicts his claim: if $F$ assigns and an assignation is a representation, then $F$ is a representation, and not the name of the representation. Therefore

\textit{... a representation ...} is an assignation \textit{...}If “$F$” designates a representation and $F$ assigns ...

is the correct re-formulation of (3.xv). Analogous mistake (his §6.7) affects

(3.xvi) \textit{we use $k$ as a name for a numeral ...} So the numerals are $k_0, k_1, ...$

since the bold symbols ought to be both the names of numerals and the numerals. To conclude, Shoenfield does not realize that the strong dialinguistic faculties of an advanced natural language allow us to speak both of an object expression and of its referent.

3.9. Two words about the autonomic homonymy, which, with arrogance and hypocrisy, is my most hated thing. A pure autonym is legitimate; every natural language with dialinguistic faculties possesses autonomic expressions as, for instance “these same words”. The dangerous passage arises when autonymy is homonymically hybridized, so that the same expression, besides being a name for itself, is also the name for something else.

Anyway I am afraid that the fight I want to put up against autonomic homonymy will add a new pearl to the collection of brilliant failures I have already recorded in my fights against arrogance and hypocrisy.

3.10. An analogous dialinguistic mistake affects the application of quantifiers. For instance Cappelen and Lepore (under the voice \textit{Quotation} in Stanford Encyclopedia od Philosophy) argue their claim

(3.xvii) \textbf{BQ2}. \textit{It is not possible to quantify into quotation}

through the following example (quotation marks replace their single inverted commas). While

(3.xviii) “bachelor” has eight letters

is a true statement, the respective existential quantification

(3.xix) $\exists x (”x” \text{ has eight letters})$

is false, since evidently “” is a one letter symbol.

The mistake affecting (3.xix) concerns the absolutely abusive quotation marks enclosing the second occurrence of the variable. Once such a variable in its first occurrence ranges over the term representing the subject which the predicate of (3.xviii) is ascribed to, in (3.xix) the same predicate must be ascribed to the variable, not to its name.

In other words. Since the universally accepted axiom for “$\exists$”-introduction is not
(3.xx) \[ P(a) \supset \exists x(P(\text{“}x\text{“})) \]

but rather (for instance Kleene 1971, §19, postulate 11)

(3.xxi) \[ P(a) \supset \exists x(P(x)) \]

the derivation of (3.xix) is wrong; on the contrary

(3.xxii) \[ \exists x(x \text{ has eight letters}) \]

is the perfectly sensible (and even true) quantification stating that a term having eight letters does exist.

Of course such a conclusion does not at all entail that (3.xix) is a statement to reject always; there are contexts where it is unobjectionable. For instance, as Coliseum is the most famous Roman amphitheatre

(3.xxiii) the name of the most famous Roman amphitheatre has eight letters

is a true statement. But the correct procedure force us

- either to quantify over names of monuments, and then (3.xviii) continues being wrong and (3.xxii) true
- or to quantify over monuments, and then (3.xix) is unobjectionable (actually a monument whose name has eight letters does exist) while (3.xxii) is wrong (no monument has eight letters, obviously).

In the latter case it is evident that the change of subject entails a change in the respective predicates (“to have eight letters” vs. “to have an eight letters name”), so that the dialinguistic import of quotation marks and the dialinguisticity of *name* compensate one another.

I do not enter into the legitiity of (3.xvii); I simply note however that the claim is not supported by the example, since (3.xix) is not the correct quantification of (3.xviii). I think that mistakes of this sort are favoured by an insufficiently sharp distinction between values and substitutors of a variable.

3.11. A conclusion imposes: a discipline which arrogates to itself the right of facing very difficult arguments before having made ready adequate mental and symbolic apparatuses, brings about its own ruin. In this sense the impudence of contemporary Logic recalls the usages of many Renaissance ladies, who paid much more attention to their making up and paludaments than to their personal hygiene. Purificatory lavacres are indispensable.

This notwithstanding, mankind seems to meet with difficulties in realizing what a great piece of luck is my arrival on the stage of Logic
CHAPTER 4
DEFINITIONS

4.1. The usual acceptation of “to define” is very wide, since we can define the rules of a play, the boundaries of a State, the powers of an office et cetera. In this sense whatever intervention adding sufficient information to identify a certain referent is a definition. Even a jaguar marking its territory is defining it. Fortunately we can avoid being involved in urological practices, since our interest is focused on definitions concerning the information adduced by a linguistic expression. This notwithstanding, under the informational approach (Suppes 1957 §8.2)

\[ A \text{ definition is a statement which establishes the meaning of an expression} \]

is too restrictive a definition of -definition-. In fact the wider viewpoint of the informational approach can easily concern definitions which are not statements. For instance the young mother uttering “hat”, “hat”, “hat” while showing different hats to her son is evidently performing a definition.

The crucial requisite is supplying the information through which a meaning is assigned to a definiendum, not the means (linguistic or ostensive) adopted in order to supply such an information. However linguistic definitions will be henceforth privileged.

4.2. Assuming (roughly)

\[ \sigma x y \]

as the general scheme of a definition, is assuming that the first variable ranges over expressions (of a given language \( L \), \( \sigma \) refers to) and the last variable ranges over meanings (a more detailed analysis in §4.6)). The informational levels involved by (4.ii) are evidently two: one for the object piece of information \( y \) (that is the piece of information which by definition is adduced by the definiendum \( x \)), and one for the piece of information adduced by the whole (4.ii), that is the piece of information telling us that (in \( L \)) the object piece of information \( y \) is adduced by the expression \( x \).

Therefore a definition is an intrinsically hyperlinguistic intervention. A due conclusion whose insidiousness is evidenced by the acritical resort to non(synonymous verbs (as “to be”, “to mean”, “to stand for”, “to be equivalent to”, et cetera), or even to different symbols (as “\( \rightarrow \)”, “\( = \)”, “\( \equiv \)”) in order to express the relation between a definiendum and its definiens.

4.2.1. In order to avoid misinterpretations, let me recall some previous assumptions.

The \( \sigma \)-relation between a sign (strictly: between the image of a sign) and the piece of information it adduces (§1.6) is intrinsically mental: no \( \sigma \)-relation without a knower.

The \( \sigma \)-relation is conventional; it depends on a code, usually the code of a public language, the linguistic ability is the mental faculty of establishing a net of (conventional) \( \sigma \)-relations.

The mental intervention necessary to abstract a type from tokens is absolutely distinct from the mental intervention necessary to associate a certain meaning to a certain sign. The latter, so to say, is the second stage of a procedure whose first stage is just the abstraction of the type (otherwise every occurrence of a token should demand a specific definition).

4.2.2. The mental character of the \( \sigma \)-relation is a claim subject to the following (and superficial) objection.

The physical contiguity between the golden

\[ \text{(4.iii)} \]

finely painted on its stern makes (4.iii) the name of this same yacht, quite independently of any mental intervention.

Reply. Their physical contiguity is nothing but an ostensive informational source telling us that actually the (image of the) name and the (image of the) yacht are connected by a \( \sigma \)-relation. Yet identifying the physical relation of contiguity between the painted name and the yacht with the semantic one, is a mistake: also the golden arabesques finely painted to enclose (4.iii) are physically contiguous to the yacht, nevertheless they do not name anything, simply because they are not interpreted as meaningful marks (i.e.: simply because we assume that their image is not the origin of any \( \sigma \)-relation).

4.3. In order to classify formal definitions let me propose a fanciful example.

By supposition the worldwide envied lady who owns the Parthenon, owns the Coliseum too. This piece of information can be indifferently adduced by

\[ \text{(4.iv)} \]

The Coliseum is owned by the owner of the Parthenon

or by

\[ \text{(4.v)} \]

The owner of the Parthenon is the owner of the Coliseum

or by

\[ \text{(4.vi)} \]

The Coliseum and the Parthenon are co-owned

(of course two or more things are co-owned iff they belong to the same owner(s)).
Nevertheless while (4.iv) is centred on a relation (“to be owned by”) between a monument and an owner, 
(4.v) is centred on a relation (“to be”) between owners, and (4.vi) is centred on a relation (“to be co-owned”) 
between monuments.

The three formulations above correspond respectively to the three strict linguistic formulations of a 
definition (where monuments become signs and owners become meanings). In fact a linguistic definition can be 
strictly formulated
- by signification (§4.4)
- by semantic identity (§4.5)
- by synonymy (§4.6)
(I speak of strict formulations because (§4.8) there are linguistic formulations whose hyperlinguistic dimension is 
not explicit).

4.4. The scheme

(4.vii) 
that is, for instance, 

(4.viii) “regular (polygon)” means *equilateral and equiangular (polygon)*

illustrates the simplest version (§4.4.3) of definitions by signification. Both the presence of “σ” and of asterisks 
make manifest the hyperlinguistic dimension of (4.vii).

The formulation by signification (which anyhow can be immediately and plainly translated into a 
formulation by semantic identity or by synonymy) plays a privileged role since it respects in the most scrupulous 
manner the task of assigning a meaning to a definiendum (4.i).

A fastidiousness. In (4.viii) “polygon” occurs between parentheses because when we do not speak of 
polygons but, say, of soldiers, “regular” does not at all mean *equilateral and equiangular*. Another fastidiousness: 
“polygon” has been and will be used as an abbreviation of “plane polygon with rectilinear sides”.

4.4.1. Since σ is a projective relation (that is a relation connecting a syntactic entity with a semantic one), a 
semantic diacritical symbol as asterisks is necessary to write down a correct formulation by signification. Two 
illustrious examples of the impasse otherwise impending are given by Pap, where he writes 

(4.ix) “fy” means “the number which is the immediate successor of y”

(1964, in Olshewsky 1969, p.287), and by Tarski, where he writes 

(4.x) It might appear ... that “true sentence” ... means nothing other than “provable theorem”

(1936, §3); they did not realize that a scheme like 

(4.xi) “x” means “y”

is either ambiguous or incoherent. In fact I remind the reader that the only way to save (4.xi) from incoherence is to 
accept quotation marks as an ambiguous symbol which in its first occurrence must be interpreted as a name of the 
enclosed expression and in its second occurrence must be interpreted as a name of the piece of information adduced 
by the enclosed expression. Yet, also leaving out of consideration that neither Pap nor Tarski point out the problem, 
the ambiguous agreement would represent however a very unprofitable way out: logic and ambiguity are (at least: 
ought to be) mortal enemies. In this sense, frankly, I think that (4.ix) and (4.x) are further evidence of the already 
denounced insufficient dialinguistic perspicuity.

Let me indulge in a frivolous analogy. Just as it would be a very hard task to put a flattened top hat on 
before the magic touch of the víveur restores its third dimension, it is a very hard task to solve enormous problems 
of logic (one for all: the problem of truth) with a diacritical symbology unable to account for their informational 
dimension.

4.4.2. The explicit formulation of (4.vii) in ordinary language is 

(4.xii) “...” σ *...-*

and (4.xii) may ring like a false note; yet this remark concerns only the elegance of the sentence. In fact, if what a 
human being writes is a writing and what a machinery produces is a product, what an expression means is a 
meaning; then (4.xii) is perfectly proper. The conclusion can be confirmed by instancing (4.xii) on an example 
where both couples of quotation marks enclose the same expression (“equilateral” means the meaning of 
“equilateral”, say): the definition becomes a mere tautology, but a tautology entails automatically it’s properness. 
Moreover if we substitute “means” by “adduces”, (4.xii) does no longer ring like a false note since the inelegant 
repetition is avoided.

4.4.3. Indeed (4.vii) is a simplified scheme. The general scheme is 

(4.xiii) “...” σL1 *...-*L2

where L1 is the language the established signification belongs to, and L2 is the language whose semantics is used to 
identify the meaning adduced by the definiens; then (4.vii) is a simplified scheme because, under the implicit 
assumption L1=L2, the indexes are omitted.
In order to write down a definition by signification, the basic problem is finding out the (three) expressions through which the identification of the (three) elements occurring in (4.ii) can be achieved. As for the definiendum, under the standard criterion (§3.6) it is sufficient to enclose the same definiendum within quotation marks. As for the relation, “σ” is the obvious solution. But as for the definiens, its identification implies the appeal to another signification (not necessarily belonging to the language of the definiendum). So for instance the understanding of (4.xiv) “casque” means in English *parte della armatura a protezione della testa* presupposes the understanding of the pieces of information adduced in Italian by “parte”, “della” et cetera.

Yet a doubt may arise as soon as (4.xv) defines -casque-is recognized as a proper (and true) proposition: in (4.xv) what specific dialinguistic symbol hyphens do stand for? The answer is clear: since (4.i) defining is establishing the meaning of an expression, hyphens stand for quotation marks. In fact if we should read (4.xv) as

(4.xiv) defines *casque*

the same definition of asterisks (§2.4) would lead us to

(4.xiv) establishes the meaning of the meaning of “casque”

that is to an improper sentence.

Therefore also in the informational approach what we define are words, not meanings.

4.4.4. Ostensive definitions can be easily interpreted as definitions by signification. For instance, in the case mentioned in § 4.1, the definiendum is uttered (“hat”, “hat”, “hat”), the meaning is inferred by an abstraction on the common connotation characterizing the various exhibited objects (roughly: all of them are covering for the head), and the σ-relation linking the uttered word with this inferred meaning results from the context. Although affected by informal factors (the hyperlinguistic intervention follows from an intuitive association) and by intrinsic limits ((how could the young mother define *otherwise* through an ostension?), ostensive definitions represent the basis of every natural semantics.

4.5. The scheme

(4.xvi) *...* = *...* 

that is, for instance,

(4.xvii) *regular (polygon)* = *equilateral and equiangular (polygon)*

illustrates the formulations by semantic identity in its simplest version. In fact (4.xvi) says that the definiendum and the definiens adduce the same piece of information (therefore “=” occurs in (4.xvi) as an unobjectionable symbol of identity). Of course

(4.xviii) the meaning of “regular (polygon)” is the meaning of “equilateral and equiangular (polygon)” is the explicit formulation of (4.xvii). The hyperlinguistic dimension of (4.xvi) and (4.xvii) results from the occurrence of asterisks, just as the hyperlinguistic dimension of (4.xviii) results from the occurrence of “meaning”.

4.5.1. Exactly as (4.xiii) is the general version of (4.vii)

(4.xix) *...* _L_1 = *...* _L_2

is the general version of (4.xvi).

4.5.2. What I stated in §4.4.1 with reference to formulations by signification can be re-proposed with reference to formulations by semantic identity, since if our endowment of diacritical symbols were limited to quotation marks, a scheme focused on the relation of identity would be either incoherent or ambiguous (§4.6.2).

4.6. The scheme

(4.xx) “...” ≡ “...”,

that is, for instance

“regular (polygon)” is semantically equivalent to “equilateral and equiangular (polygon)” Illustrates the definitions by synonymy in its simplest version. In (4.xx) “≡” is a symbol of equivalence and the index “σ” tells us that the equivalence concerns the semantic dimension (a synonymy is nothing but a semantic equivalence). So, since inverted arrows have been introduced as a symbol of synonymy (§2.10.1),

(4.xxi) “...” ⊳ “...”

is the alternative formulation of (4.xx) which will be privileged. Here I do not dwell on the relations of equivalence, which will be analyzed in the course.

4.6.1. I recall §4.4.3 and §4.5.1 to underline that (4.xxi) is a simplified scheme; in fact

(4.xxii) “...” _L_1 ≅ _L_2 “...”,

...
is the general scheme applicable where the two expressions belong to different linguistic codes. Yet, for sake of concision, wherever possible I will reason on the simplified schemata (which obviously hold in formal systems, where we act within an only language).

Once we agree that in its technical acceptation an abbreviation does not imply any shortening component, we can also call “by abbreviation” this kind of definitions; this notwithstanding, though the definiendum of a definition by abbreviation may also be longer than its definiens, it is usually shorter (mainly because if we already have a shorter expression for a meaning, it would be an anti-economic intervention to introduce a longer one).

4.6.2. In the current treatments, strict formulations as (4.xiii) and (4.xxii) are often replaced by (4.xxiii) “...,“ = “.,.“
that is by a hybrid scheme which, once more, compels celebrated authors to untenable devices. In fact a point is sure: except the trivial case of a tautology, the two expressions enclosed within quotation marks in a particularization of (4.xxiii) are different, therefore the identity symbol “...” cannot be read as an identity symbol. Actually, for instance, Carnap reads it as “to be interchangeable with”, and Church (in Runes) as “to stand for”. But consequently and evidently, since elsewhere “...” continues adducing identity, these readings make it an ambiguous symbol and, moreover, an ambiguous symbol whose secondary meaning ought to be formally defined. On the contrary
- in (4.xvi) “.” occurs properly, because in (4.xvi) it actually adduces identity (of meaning)
- in (4.xix) “,” can properly be read as “to be interchangeable with” or as “to stand for” or as “to be an abbreviation of” because, once asterisks have been introduced in the formal system, such readings are exactly entailed by the definition of inverted arrows (§2.10.1).

4.6.3. An important note. The fact that in (4.xxi) neither asterisks nor “σ” occur does not at all mean that, therefore, a definition by synonymy, far from being hyperlinguistic, is simply metalinguistic; the hyperlinguistic dimension of (4.xxi) hides in inverted arrows (§2.10). Accepting the reciprocal substitutability of two signs without imposing the identity of meaning (in the same interpretation) would be a ruinous step for the consistence of whatsoever theory. In order to support this conclusion it is sufficient to evoke the admissibility criterion (Salmon 1966 p. 64 quoted in Hajek 2003, §2): the meanings assigned to the primitive terms ... transform the formal axioms, and consequently all the theorems, into true statements. In other words: the risk connected with Carnap’s and Church’s position is not to realize that *to be interchangeable* or *to stand for* mask under their syntactic pertinence a semantic regard.

Let me insist. In a formal system, definitions are simply the means to introduce non-primitive symbols as abbreviations of primitive or previously defined symbols; therefore the strict formulation of such definitions is (4.xxi). But as soon as we interpret “symbol” (or “sign”) in compliance with its strict acceptation, the semantic regard becomes evident, since a symbol is something which belongs to a language, and a language is intrinsically an instrument to communicate information. Thus, though a formal system leaves any specific interpretation out of consideration, two expressions, in order to be reciprocally substitutable, must adduce the same meaning; otherwise the admissibility criterion might be respected by one of them and violated by the other one. Shortly: in whatever formal theory whatever abbreviation entails an interpretative equivalence.

The understanding of this claim might be helped by reconsidering (4.vi); it speaks of monuments, but the involved (cadastral) relation is ‘semantic’, as it implies the figure of an owner. In this sense, as soon as we realize that in *sign* the semantic dimension is essential, we realize that “to be synonymous” expresses a relation of semantic regard in spite of its syntactic pertinence.

4.7. Let me spend two lines apropos of a coarse puzzle concerning definitions. On the one hand a definition adduces a new piece of information (without it, obviously, we could not manage the definiendum); on the other hand since (I am re-quoting Suppes) a new definition does not permit the proof of relationship among the old symbols which were previously unprovable, the same definition should not adduce any new piece of information.

In order to solve the puzzle it is sufficient to mind the distinction between the object formal language the definiendum belongs to and its ‘metalanguage’ (hyperlanguage) the definition belongs to. I repeat: the piece of information adduced by a definition is a ‘meta-information’ (a hyper-information) concerning the assignation of an object information to a definiendum. A situation analogous to a direction teaching us how to consult a telephone book; it explains how to read the list of the users, yet it does not enrich the same list.

4.7.1. This dialinguistic distinction (also: this projective distinction) between the planes of the definiendum and of the definition explains immediately why the syntactic status of the definiendum does not influence the syntactic well formation of a formal definition. For the sake of concision I limit to definitions by signification a discourse which can be immediately referred to definitions by semantic identity or by synonymy). For instance

(4.xxiv) by definition “femur” means *thigh bone*
(4.xxv) by definition “to susurrate” means *to speak softly*
(4.xxvi) by definition “otherwise” means *in a different manner or by other means*
are syntactically well-formed hyperlinguistic sentences concerning respectively a substantive, a verb and an adverb (of the object language). As we already know (§2.5), the syntactical well formation of (4.xxiv), (4.xxv) and (4.xxvi), notwithstanding the different syntactical status of the three definienda, depends on the fact that quotation marks and asterisks are dialinguistic symbols substantivizing (in the dialanguage) whatever expression (of the object language) they enclose.

From this viewpoint the distinction between explicit and contextual definitions (for instance Pap 1964 in Olshewsky 1969, p.285) is of no theoretical moment, since we can consider all definitions as contextual and remark that there are contexts where some syntactical simplification is attainable. In particular the distinction emphasized by Pap’s examples on -brother- is influenced by the double reading discussed in §19.7 (the brotherhood function, so to write, is not its generic ‘value’).

4.8. A traditional (yet naive) distinction opposes nominal and real definitions (that is, in Ockham’s terminology, *definitiones quid nominis* and *quid rei*). This distinction is currently explained by statements like

> Real definition is distinguished from nominal definition as being the definition of a thing rather than of a notation

(Church under the voice “Definition” in Encyclopædia Britannica 1963) or like

> A real definition is understood to define the object for which a term is used, a nominal definition to define the term …

(Olshewsky 1969, footnote p.280). I dissent because of a basic difficulty: what on earth do *the definition of a thing* mean? Once any metaphorical interpretation under which, say, “to define” becomes a synonym of “to make” or of “to design” is banned,

> to define a hat

is an improper expression. Logicians are not jaguars, but neither hatters; what they can define is the informational import of a word, not the shape of a garment.

Let me apply Carnap’s well known distinction between material and formal modes of speech to a topic where it is actually enlightening, that is to definitions. Then

(4.xxvii) a pentagon is a polygon with five sides

is the material version and

(4.xxviii) “pentagon” means *polygon with five sides*

et cetera are the formal versions of the same definition. But what we can introduce in a formal system are words, not geometric figures: (4.xxvii) is nothing but a pseudo-objective way to express the piece of information adduced by (4.xxviii) (or by (4.xxix) et cetera). In my opinion ‘real definitions’ are nothing but definitions formulated in the material mode of speech.

4.8.1. Incidentally. So far as his original examples are concerned (1935, 7), Carnap’s distinction is untenable. To read

(4.xxix) This book treats of Africa

as a pseudo objective sentence for the formally correct

(4.xxx) This book contains the word “Africa”

is a manifest far-fetched claim: *being treated in this book* is a connotation (*a quality*) of Africa exactly as *being visited by Mr. A*.

Furthermore, while the alethic values of (4.xxvii) and (4.xxviii) must be the same, the alethic values of (4.xxix) and (4.xxx) may be different.

The informational approach is also fit for avoiding another analogous mistake. In fact he claims that

(4.xxxi) (the words) “Morning-star” and “Evening-star” are synonymous

is the formal mode of speech through which we can translate the pseudo object sentence

the Morning-star is the Evening-star

(material mode of speech). I dissent as I think that his position does not account for a basic informational discrepancy. In fact, for instance, as soon as we compare (4.xxxi) with

(4.xxxii) (the words) “femur” and “thigh bone” are synonymous

we can realize that while in (4.xxxii) the two mentioned expressions refer to the same informational nucleus (to the same image, so to say briefly), in (4.xxxi) they do not, since the piece of information we draw from the sight of the Morning-star is absolutely different from the piece of information we draw from the sight of the Evening-star. In other words. While, until the discovery of their identity, “Morning star” and “Evening star” named two different heavenly bodies, never “femur” and “thigh bone” named two different bones; in this sense they are connected by a very synonymy.

As soon as we realize that a third protagonist (the meaning) does exist between sign and referent, statements of identity are no longer puzzling. Anyhow I shall retake this crucial passage.
4.8.2. Ajdukiewicz (1958) writes: there is no general concept of definition of which the concepts of real definition and nominal definition would be specifications; the word “definition” has in isolation no meaning at all. Indeed, on the ground of the proposed considerations, Ajdukiewicz’s claim seems to me too hazardous to be tenable (all the tenable hazardous claims I know are exactly mine). The word “definition” has in isolation a precise meaning since a definition is any intervention supplying the piece of information necessary to assign a meaning to a definiendum, quite independently on the means through which such a piece of information is supplied (let me evoke again the mother uttering “hat”, “hat”, “hat”). His too hazardous claim is born by the lacking informational dimension inducing him to over-estimate unessential peculiarities.

4.9. Definitions are actually classified through paradigms inspired by numerous and heterogeneous criteria. Yet many of these criteria can be neglected because they are superficial enough to result of no theoretical interest.

A current intriguing criterion (for instance Wikipedia, under the same voice) opposes descriptive definitions (which refer to the general use) to stipulative ones (which refer to the speaker’s immediate intentional meaning). So, while (4.xxvii) instances a descriptive definition (for the notion of a pentagon belongs to the general use), (4.xxxiii) a pentahusband is a man with five wives
instances a stipulative definition (for, as far as I know, the notion of a pentahusband has just been introduced by my personal intervention, that is by (4.xxxiii)). Of course, in spite of its vagueness, the opposition between descriptive and stipulative definitions concerns different informational situations and as such it deserves a more systematic analysis. For the sake of concision, henceforth I only consider definitions by signification.

4.10. Let me scrutinize the matter from the viewpoint of the interpreter. The piece of information he draws from a given definition, that is from the acquirement of the signification through which a certain meaning is assigned to a certain definiendum, depends on what the same interpreter previously knows about such a signification (two different interpreters or the same interpreter in different moments may draw different pieces of information from the same definition). If we schematically and propaedeutically assume that these possible cognitive situations are only two (*known* vs. *unknown*), such situations are four, and precisely
- both definiendum and meaning are previously unknown (total definitions)
- the only meaning is previously known (christening definitions)
- the only definiendum is previously known (connotative definitions)
- both definiendum and meaning are previously known (heuristic definitions).

Indeed under the mentioned assumption (*known* vs. *unknown*) heuristic definitions are border-line cases. In fact if both correlata $x$ and $y$ are previously known, there is no informational gap to fill (the definition does not increase the statute). Yet as soon as we refine the opposition, so accounting for the thousand degrees of our actual knowledge about a topic, we can reasonably refer heuristic definitions to cognitive situations where a previously gross signification is better specified (§4.12).

4.10.1. The basic achievement is that, according to the above approach, the status of a definition varies with the statute of the interpreter; an unquestionable relativization, yet, because what can be a total definition for Plato, say, may be a christening definition for Socrates.

Getting over this relativization by making reference to the (indeed vague) figure of an average interpreter (whose statute corresponds just to the current knowledge about the topic involved by the definition under scrutiny) is restoring the (indeed vague) opposition between descriptive and stipulative definitions.

4.11. In order to emphasize the dependence of the classification on the statute of reference, in the first example, instead of considering different interpreters, I consider the same interpreter in different cognitive situations. So let (4.xxxiv) “duagon” means *(regular) polygon with two sides*
be the example under scrutiny (incidentally: the parentheses of (4.xxxiv) say that the specification of regularity is superfluous because, as we shall see, irregular polygons with two sides do not exist).

4.11.1. Total definition. Plato never met “duagon” and never minded the notion of a polygon with two sides, then (4.xxxiv) defines a signification concerning a previously unknown term and a previously unknown meaning.

4.11.2. Christening definition. Plato was previously convinced that Euclide’s notion of a polygon was too restrictive, since the case of $n=2$ is neither considered; but the same Plato never coined a name for this new geometrical figure, nor met such a name before facing (4.xxxiv), which then assigns a previously unknown name to a previously known notion.

4.11.3. Connotative definition. The advanced essay on plane geometry Plato is studying speaks of duagons without any explanation.
4.11.4. Heuristic definition. Plato replies:
- But what on earth is a regular polygon with two sides?

And the master:
- Have you a clear idea of a polygon with three or more sides?
  - Yes, I have. A polygon is a plane closed figure bounded by straight sides, so that if we start from a vertex and run along its perimeter, we come back to the point we started from.
- Do you agree that a polygon with \( n \geq 2 \) sides is regular iff it is equiangular and equilateral?

And Plato, after a quick reflection
- Yes, I do.
- Do you agree that the common value \( \theta \) of its \( n \) internal angles is \((n - 2)180°/n\)?

And Plato, after meticulous computations
- Yes, I do.
- Do you agree that, therefore, in a duagon the internal angles must be two, both of \(0°\)?
  - Yes, I do.
  - Do you agree that, once fixed a point as the starting vertex and an arbitrary segment as the first side, to follow with a \(0°\) internal angle is to come back towards the starting vertex?
  - Yes, I do.
  - Do you agree that a segment in going and coming back binds a closed plane figure (although with a null surface)?
    - Yes, I do.
    - Do you agree that such a figure satisfies all the mentioned requisites of a regular polygon?
      - Yes, I do.
      - Therefore you have a precise geometrical idea of what a regular polygon with two sides is. And indeed it complies too with our gross intuition according to which a regular polygon with \( n \) sides is the most regular polygonal arrangement of a regular polygon with \( n + 1 \) sides once we erase one of its sides.
      - But are we sure that also the other quantities characterizing the notion of a regular polygon apply as well to a duagon?
  - As far as I know, they do. For instance the notions of inscribed and circumscribed circumferences survive respectively in the middle point of the segment and in the circumference admitting it as a diameter. For instance the sum of its external angles (720°) continues satisfying the general formula \( n(360°-\theta) \). And so on.
  - I understand: a regular polygon with two sides is a couple of identical and superimposed segments.

4.12. Besides the Socratic maieutics, heuristic definitions play a momentous role in the history of the human culture because many fundamental philosophic problems can be conceived as the research for an heuristic definition. The best example is Tarski’s research for a definition (definition, I emphasize) of “truth”. Awaiting it (Chapter 5), I can remind St. Augustine’s reflection about *time*, or, even better, Frege’s studies on a satisfying definition (definition, I re-emphasize) of “number”. Also before the publication of his *Die Grundlagen der Arithmetik*, “number” and *number* were respectively a word and a notion of consolidated use (thence the heuristic character of the pursued definition), yet any attempt failed to answer the question: what is a number? For instance, the purpose of the mathematician who claimed that “number” was nothing but a synonym of “difference” was to find out and display the connotation that, although hidden in the semantic tissue of ordinary language, is essential in *number*. And in order to understand that his claim is untenable it is sufficient to remember the *bon-vivant* who would be struck with horror at the idea that his “Vive la difference!” should be interpreted as an ode to the number.

4.13. Let me insist through an example concerning a translation dictionary (in its two sections English-Italian and Italian-English, say). The manifest ellipticity of a formulation like (4.xxxv) **meddlesome** invadente

is inconsequential because, since he who consults a dictionary is previously aware of the context (is previously aware that he is dealing with definitions), simplified graphic conventions are legitimate. And exactly the ellipticity of (4.xxxv) allows its reading as a definition by signification, or indifferently by semantic identity, or by synonymy; these readings follow directly from the interpretation of the mentioned graphic conventions.

In all cases the same title of the volume under consultation (“English Italian Dictionary”, say), through its first word tells us that the first word of (4.xxxv) belongs to the lexicon of English, therefore that *LI* is English, and through its second word tells us that *L2* is Italian. Under a reading by signification (I evoke (4.xiii)) the boldface acts as a couple of inverted commas, the following space acts as a “\( \sigma \)”, the Roman type acts as a pair of asterisks.

On the contrary, under a reading by semantic identity, both boldface and Roman type act as asterisks, the space between them acts as a predication of (semantic) identity and so on.
Now I analyze (4.xxxv) according to the cognitive endowment of four different readers.

4.13.1. First reader. My deeply uncultured Italian friend Jim, glancing inattentively through the English-Italian section, meets casually (4.xxxv); thus he learns that this previously unknown “meddlesome” means in English the previously unknown *invadente*, and fills his latter informational lack by asking me what “invadente” means in Italian. For Jim (4.xxxv) is then a total definition.

4.13.2. Second reader. My semi-cultured English friend Ted, who knows the meaning of “meddlesome”, consults that voice to learn the Italian adjective such a meaning is adduced by. For Ted (4.xxxv) is then a christening definition telling him that “invadente” is the Italian adjective he is searching for.

4.13.3. Third reader. My semi-cultured Italian friend Bob meets “meddlesome” while he is reading an English novel and, since he does not know its meaning (since he does not know *meddlesome*), he brings himself to consult the voice “meddlesome” in the dictionary. For Bob (4.xxxv) is then a connotative definition telling him that the well known *invadente* is the meaning to correlate with the well known “meddlesome” he was starting from.

4.13.4. Fourth reader. My superficially polyglot friend Tom knows vaguely that *meddlesome* is a rather negative connotation, then he consults the dictionary just to improve this fuzzy signification. For Tom (4.xxxv) is then a heuristic definition.

4.13.5. Concisely. The English-Italian section, normally, is a book of christening definitions when in English hands, and a book of connotative definitions when in Italian hands. Vice versa, obviously, as for the Italian-English section.

4.14. Another interesting and non-fictitious example concerns *planet*. The four contexts can be clearly singled out, as ancient astronomers observed that seven heavenly bodies were in apparent wandering motion with respect to the fixed stars, and agreed to call them “planets” (“πλαναω” “I wander”), but just in our days a congress of astronomers is discussing the precise connotations a heavenly body must possess in order to be classified as a planet, While this congress is pursuing a heuristic definition, the absolutely incompetent Plato who, reading an astronomical report, asks Socrates “What is a planet?” is pursuing a connotative definition et cetera.

4.15. The tripartition of §4.3 and the tetrapartition of §4.10 are completely independent. The latter can be applied to definitions by semantic identity or by synonymy, but even to ostensive definitions. For instance the mother uttering “hat” is performing
- a total definition if the child ignores both the notion of a covering for the head and the uttered word
- a christening definition if her son already knows the notion but ignores the word,
- a connotative definition if the child asked her what is a hat (he knows the word but ignores its meaning)
- a heuristic definition if the child is learning that there are covering for the head which are not hats, but caps, helmets et cetera.
5.1. I call “alethics” (from “αλεθεια”) the doctrine studying the matter of truth, once *matter of truth* is assumed in the wide acceptation according to which, for instance, also the notions of decidability, probability, falsity concern the matter of truth. This notwithstanding, for the sake of concision, everywhere possible I will reason only on *true*, entrusting the reader with the immediate extrapolations.

Consequently I call “alesis” a statement where an alethic attribute is ascribed to an object statement.

Why a new word? To fill a lexical gap. “False”. “Decidable” et cetera adduce alethic attributes (values) just as “female”, “hermaphrodite” et cetera adduce sexual attributes (values). Without the new word, for instance, we should say that “false” adduces a truth attribute, and this seems to me a para-oxymoron like saying that “female” adduces an attribute of masculinity. Furthermore every important doctrine has its specific name, and for sure alethics is an important one.

5.2. The only aim of this chapter is to show that the question

(5.i) are sentences or propositions the very objects of truth?

admits only one consistent answer: propositions. In subsequent chapters alethics will be faced formally through an axiomatic system where deductive logic becomes simply a border case of inductive one.

5.2.1. Indeed, at first sight, (5.i) seems a pseudoproblem. Already Parsons (1974, Note p.407) had written:

*I assume that the primary truth vehicles are sentences; otherwise “is true” should be read as “expresses a true proposition”. And Gupta (1982, p.4): I assume that the objects of truth are sentences….. . A more intensional proposal, such as that the objects of truth are … propositions, is not acceptable to us …. In any case a theorist who insists that it is the latter that are the objects of truth may take us to be giving a theory not of truth but rather of the concept “being a sentence that expresses a true … proposition”.

On these grounds their basic argument can be extrapolated as follows: these who insist that propositions are the objects of truth can define “veracious” (“fallacious” et cetera) to mean *expressing a true (a false et cetera) proposition* and refer the theory to veraciousness (fallaciousness et cetera).

I call such an argument “pertinence trick” to mean simply that in my opinion it is intrinsically misleading.

5.3. The current canonical position, according to which sentences are the actual objects of truth, follows from a famous article (1943), where Tarski, inquiring into a definition (definition, I re(emphasize) of truth, claims that his purpose is not to assign a new meaning to a familiar word, but, on the contrary, to understand its current meaning. I agree totally with his purpose: the task is not to dictate a sovereign (stipulative) definition, but to reach an heuristic one. Then both of us agree that

(5.ii) when we currently say that something is true what are we speaking of?

is an equivalent formulation of (5.i).

Tarski remarks that “true” is used with reference sometimes to sentences, and sometimes to propositions, yet he concludes that for many reasons the former choice represents the best one, also because “proposition” is a term whose meaning has been very debated but never satisfactorily clarified (certain ideal entities called “propositions”). Since he does not expose the many reasons supporting his choice I cannot discuss them; anyhow I can expose some very strong arguments supporting my claim according to which sentences, far from being the best choice, are even an incoherent one.

5.4. The first argument is based on a rule I call “criterion of influence”. By

no predicate can concern a kind of referent whose characteristics
do not influence the alethic value of the respective message

I epitomize a propaedeutic version of this criterion, whose obviousness is promptly showed through some minute examples.

5.4.1. Bob shows us a one dollar banknote and says

(5.iii) it is a little bill, indeed

thus puzzling us: is he speaking ‘syntactically’ of its small size or is he speaking ‘semantically’ of its poor purchasing power? In other words: does his “little”, which surely pertains to the banknote, regard the banknote itself or rather its purchasing power? Of course, as logicians, we cannot follow the simplest procedure (to submit directly our perplexity to Bob). Yet Tom is enlightened by a genial idea:

is this one little too?

he asks Bob showing him a 1000$ banknote. If Bob answers
we all understand that (5.iii) speaks of the size; on the contrary, if Bob answers no, obviously!

we all understand that (5.iii) speaks of the purchasing power. Why? Because, since the two banknotes have the same size and a very different purchasing power, if a change involving only the purchasing power overturns Bob’s opinion, surely he was not speaking of the size (as to the size, nothing is changed). Et cetera.

5.4.2. Another example (§3.2.2) is the lady’s cross-word. Her

-shuffle- is too short

cannot but refer to the word intended as a graphic type because any modification preserving the same graphic type (any modification of the phonetics or of the graphic token) is of no moment as for the alethic value of (5.iv).

5.4.3. The criterion of influence, above applied to -is little- and to -is too short-, can be identically applied to any alethic predicate (that is to any alesis). On its basis it is sufficient to ascertain whether an alesis, so to say, is homonymy-independent or synonymy-independent. An easy task indeed, since we can ascertain both the homonymy dependence and the synonymy independence of an alesis. In fact, since in a situation where different sentences adduce the same proposition the alethic value must be one only, while in the reciprocal situation (only one sentence adducing different propositions) the alethic values may be different, alethic predicates cannot regard sentences and can regard propositions.

5.4.4. Let me recall the example of §1.9.2. Bob wishes a new female swan to marry his cob and a new instrument for ink in his drawings, yet, since he cannot meet both expenses, he renounces the latter. Then, since

(5.v) Bob bought a pen

is only one sentence, should we insist on the syntactical pertinence of alethic predicates, we ought to conclude that (4.v) is at the same time true (under the zoological interpretation of “pen”) and false (under the other one), so falling into incoherence. On the contrary if we acknowledge that alethic predicates pertain to propositions, all is right: since “pen” is a homonymy bearer, (4.v) adduces (at least) two different propositions, and while one of them is true, the other is false.

In other words. The truth (or falsity et cetera) of

(5.vi) A female swan has been purchased by Bob

can be inferred from the truth (or falsity et cetera) of (5.v) in its zoological interpretation; and vice versa. But what (5.v) in its zoological interpretation and (5.vi) have in common is the piece of information they adduce, surely not the linguistic vehicle adducing such a piece of information. And just because the piece of information is the same, the alethic value must be the same.

5.4.5. Another example. Here we have two documents: the (ideographic) report of the Japanese Intelligence about the secret nuclear programs of an Asian State, and its English translation. Once agreed by hypothesis that the translation is perfect (contents-conservative) we cannot coherently assign different alethic values to the two reports, though their texts are radically different. And why is it necessary to suppose that the translation is perfect? Merely because it it were not, a discrepancy in the respective contents (meanings, propositions) could justify the assignation of two different alethic values.

5.5. The argument is immediately extrapolable from informational identities to informational implications. For instance it would be manifestly incoherent to state that

Bill is an Irish setter

is true and that

Bill is a dog

is false just because *Irish setter* implies *dog*, and this meaning dependence entails an alethic dependence.

Reciprocally if we know that the truth of

This animal is a celep

implies the falsity of

This animal is a bird

we do not need to know exactly what a celep is to conclude that *celep* implies *non-bird*.

5.5.1. Of course we cannot forbid Tarski and his epigones to agree that a sentence is veracious iff it adduces a true proposition. Actually heuristic definitions allow the introduction of attributes whose pertinence depends on our free choices. This notwithstanding the regard of such attributes is not at all free; and so then assimilating the use of “veracious” to the current use of “true” is radically untenable. We come back to Bob’s cuckoldries (§2.9). A theory of conjugal honesty grounded on a definition as “cuckold” means *husband of an unfaithful wife*
is an only apparent ‘husbandization’ of the matter: undoubtedly husbands are the objects of cuckholdness, but to dog husbands would be a stupid measure just because the only way to ascertain if a husband is a cuckold is to ascertain if his wife is unfaithful. Therefore we fall again into a situation dealing with the behaviour of a wife.

Analogously, since we have ascertained that in its current use *true* necessarily concerns propositions, a theory of veraciousness grounded on the assumption of “veracious” as an abbreviation of “expressing a true proposition” is an only apparent ‘syntactization’ of the matter. For instance, we can assign an alethic value to “Ava loves Bob or Tom” is a disjunctive sentence on the basis of merely syntactic considerations (roughly: a sentence is disjunctive iff an “or” occurs); but in order to assign an alethic value to "Ava loves Bob or Tom” is a veracious sentence we must ascertain whether Ava loves Bob or Tom, then whether *Ava loves Bob or Tom* is true. And exactly because scrutinizing the sentence without looking at the proposition would be a hopeless procedure, the appeal to “veracious” is a mere artifice; such an attribute pertains to sentences, but unavoidably regards propositions. Here is the pertinence trick.

5.6. In subsequent chapters all these intuitive conclusions will be formally derived from a strictly axiomatic definition of *true* (*false* et cetera). However they represent a massive proof that when in our everyday language we say that something is true (or false et cetera) we (necessarily though perhaps unconsciously) are referring to pieces of information. I write “pieces of information” instead of “propositions” in order to point out that *sentence*) in order to admit also non strictly linguistic transmissions of signs, the consequently widened sentences or propositions?

is reductive, since it tacitly presupposes a linguistic component while, on the contrary, alethic predicates can also be ascribed to non-linguistic objects, that is to non-linguistically mediated pieces of information.

A proposition is a piece of information aduced by a sentence, where a sentence is a sequence of words concatenated in compliance with well specifiable rules of formation. But even if we widen *language* (then *sentence*) in order to admit also non strictly linguistic transmissions of signs, the consequently widened *proposition* is still insufficient to contain the pieces of information acquired without any intervention of signs. Therefore, since alethic predicates can be properly ascribed to such pieces of information too, it would be reductive to claim that propositions are the very object of truth. The very objects of truth are pieces of information, and propositions are only a subset (the linguistically mediated pieces of information).

5.6.1. An example. Immersed in a thick fog and escorted by my mastiff Tabù I am going through an unknown vineyard. Suddenly we realize that a gigantic, anthropomorphic and hostile figure is bobbing very close to us. While my heart goes mad, Tabù growls darkly. Two ridiculous reactions, because a bee-master’s overalls hanging from a branch and tossed by a gust of wind can neither represent a danger, nor can be intimidated by a dark growl. Though (5.vii) a gigantic, anthropomorphic and hostile figure is threatening us is a reasonable way to express in English the conviction which caused my reaction, I firmly believe that no linguistic mediation intervened in the process, also because finding acceptable words for describing the experience would have requested much more time than the flashing start of my tachycardia. And anyhow it is obvious that no linguistic mediation is hypothesizable in a dog, particularly in a dog whose intelligence is rather torpid.

Then, the immediate and ailing information which finds in something like (5.vii) its linguistic expression was false (actually our reactions are ridiculous because dictated by a quite erroneous interpretation of certain sensorial perceptions). Neither are the overalls a sign, as the bee-master who hanged it on the branch did not wish to communicate anything to anybody. This notwithstanding I am quite legitimated to say that I mistook myself, then that the piece of information which finds in (5.vii) its linguistic formulation was already false before finding such a formulation; but thus I am ascribing an alethic predicate to a non-propositional piece of information.

More about this matter in §13.8.1.

5.6.2. In this essay we shall only deal with linguistic sentences and consequently with pieces of information whose propositional trait is assured. Yet these considerations about non-propositional pieces of information are not superfluous, since they show that the informational approach looks at the whole matter from a higher viewpoint. I mean that while the occurrence of a sign is necessary to configure a sentence (in its widest acceptation of a linguistic informational source), no occurrence of a sign is necessary to configure the source of an alethic object.

Hence a concise argument supporting the impossibility to give the syntactical solution to (5.ii): logic was born millions years before any language therefore, either logic does not concern sentences, or through millions of years it existed without any concern.

Once clarified this point, the mentioned linguistic character of the matter under scrutiny induces me to continue speaking of propositions to mean generically *pieces of information*.
5.7. A classical argument against the claim that sentences are the objects of truth is based on indexicality. According to it, for instance, since the alethic value of a message like (5.viii) yesterday it was raining here varies with the moment “yesterday” and the place “here” refer to, it cannot concern the sentence, which is always the same. I obviously agree with the conclusion, yet I think that the argument does not grip. Very briefly (awaiting the detailed analysis proposed in Chapter 16). Indexical terms are contextual variables, that is variables whose contingent values are fixed by the context in which they are uttered; therefore, since (5.viii) is an open sentence (a free-variables-laden sentence), the adduced (open) proposition can be alethically judged only after its closure (after the promotion of its two indexical variables). In this sense (5.viii) is a scheme of various free-variable-free sentences, and the fact that different free-variable-free sentences have potentially different alethic values is not evidence against the syntactic regard of alethic attributes.
6.1. In order to achieve an axiomatization of the informational approach in the easiest way, I begin by reasoning informally under some schematizations.

Though in our everyday practice a reliability degree (from 0 to 1, say) can be associated to any piece of information belonging to our knowledge, for the moment we only consider the extreme values, thus sharply opposing *known* to *unknown*.

For instance let me consider three consecutive tosses of a well-balanced coin. Since any outcome is either head (A) or tail (B); the possibility space \( \Omega \) of such a sequence contains eight alternatives (either AAA or AAB or ABA or ABB or BAA or BAB or BBA or BBB).

I call “basic statute” the piece of information \( k^0 \) defining \( \Omega \) and “acquirement” any subsequent piece of information \( k' \) concerning the same \( \Omega \). Simple but general evidence is that any acquirement corresponds to the preclusion of some alternative (and vice versa). For instance to learn that two consecutive but not better specified tosses gave the same but not better specified outcome precludes ABA and BAB. I call “free” the non-precluded alternatives and “statute (of g at t about \( \Omega \))” the conjunction \( k=k^0 \& k' \), that is the whole information about \( \Omega \) possessed by the knower \( g \) at the moment \( t \).

6.2. In order not to waste time with analyses of negligible interest, the coherence of the statute is presupposed; this only means that the eventual incoherence of a statute must be explicitly remarked.

The existence of a statute does not entail the ontological reality of the universe it describes. Since both information and imagination are mental things, we can speak of Plato (whose historical existence is sure) exactly as we can speak of Homer (whose historical existence is controversial) or of Polyphemus. Analogously, in order to reason on our \( \Omega \), it is not at all necessary to perform materially the tosses.

Of course a radical difference distinguishes the actual world from fictitious ones, giving the former a quite privileged role; yet this topic will be analysed in Chapter 13.

6.3. The informational approach is strongly helped by the intensional representation \( \otimes \) whose main lines I am about to illustrate.

6.3.1. Figure 6.1

![Figure 6.1]

illustrates the first basic idea of \( \otimes \): representing \( k^0 \) by a circle partitioned in as many sectors as the alternatives of the respective possibility space \( \Omega \). so, with reference to the example above, the sector 1 represents AAA et cetera, till the sector 8 for BBB.

For the sake of concision “sector representing the outcome so and so” can be abbreviated in “sector so and so” and “outcome represented by the sector so and so” in “outcome so and so”. These terminological licenses are not affected by any reasonable risk of interpretative confusions.

6.3.1.1. Incidentally. I choose a circle to recall Venn’s diagrams, yet, if the alternatives were numerous, dealing with very narrow sectors would be an annoying practice; then a more general convention could represent \( k^0 \) with a rectangle of convenient size, partitioned in the necessary number of sub-rectangles.

6.3.2. The second basic idea of \( \otimes \) is representing a piece of information \( h \) (for the moment \( h \) can indifferently be an acquirement or an hypothesis) by shading the sectors respectively precluded. Each of the \( 2^n \) different pieces of information concerning a possibility space of \( n \) alternatives can thus be unequivocally represented. For instance the piece of information that a not better specified outcome is \( B \) is represented by shading
the sector 1 (obviously if an unspecified outcome is B, every sequence is possible except $AAA$); analogously the piece of information

(6.i) the first outcome is $B$

is represented by shading the sectors 1, 2, 3 and 4 because, of course, (6.i) precludes the four possible sequences whose first outcome is $A$.

I call “virgin” any non shaded sector and “virgin field” the union of the virgin sectors. So, for instance, the virgin field representing (6.i) is the lower semicircle.

The intensionality of $\oplus$ results from the above explained ideas. Since its space does not represent sets of individuals, but pieces of information, any increment of information is represented by an increment of the shaded field, therefore by a decrement of the virgin field. In this sense the virgin field represents the residual margin of uncertainty.

6.3.3. Exactly in order to comply with this approach the third basic idea of $\oplus$ is representing any alternative by a sector whose area depends on the characteristics of the represented universe, that is, roughly speaking, on the probability of the respective alternative. I make myself clearer. Until now I reasoned under a tacit assumption: the equiprobability of the various outcomes (here is the reason why I specified that the coin is well-balanced). And actually Figure 6.1 is suited to represent also different operative contexts, provided that the octo-partition of the respective possibility spaces be uniform. For instance, so that Figure 6.1 can correctly represent the results of a well balanced mini-roulette with eight numbers, the condition must hold that the eight boxes have the same size. A well balanced mini-roulette where the numbers 1 and 2 have a box of $80^\circ$, the numbers 3, 4, 5 and 6 have a box of $40^\circ$, the numbers 7 and 8 have a box of $20^\circ$, finds its correct (even iconic) representation in Figure 6.2.

6.4. I call “measure (of $h$ under $k^o$)”, and I symbolize by

(6.ii) $\mu_k(h)$

the quantity represented by the area of the $h&k^o$-virgin field. Of course to speak of a triple $\langle k^o,h,\mu \rangle$ would be a more canonical (but, in my opinion, an intuitively less adequate) expressible way. So, for instance, with reference to an octo-partitioned mini roulette, the measure of

(6.iii) the outcome is $<5$

is represented by shading the inferior semicircle if we refer to the $k^o$ of Figure 6.1, while it is represented by shading the last third of the circle if we refer to the $k^o$ of Figure 6.2; in fact as (6.iii) precludes the alternatives 5, 6, 7 and 8, it is represented by shading such sectors, thus leaving virgin the sectors 1, 2, 3 and 4.

The criteria ruling the assignation of a measure to the various alternatives of a possibility space will be studied in due course; here I only introduce the notion, emphasizing its intrinsically relational character. In fact the above example shows that the measure of (6.iii) depends also on $k^o$ (that is, in the case of the mini-roulette, also on the sizes of its eight boxes).

I advance that uniform partitions will be privileged; yet I underline that this agreement is not an artful way for intruding a sort of equiprobability; it is only a way for avoiding mathematic and geometric complications of no theoretical moment (a uniform partition is anyhow a partition).

6.4.1. Precisely as (6.ii) symbolizes the measure of the piece of information $h$ under the statute $k^o$,

(6.iv) $\mu_{k^o&h}(h_2&h_3)$ symbolizes the measure of the piece of information $h_2&h_3$ under the statute $k^o&h_1$ that is under the statute increased by adding the piece of information $h_1$ to $k^o$. For instance, with reference to the $k^o$ of our mini-roulette, once $h_1$ precludes the even numbers, $h_2$ is $\sim 1$ and $h_3$ is $<5$, (6.iv) is the measure of the outcome 3.

6.4.2. Obviously an actual measure corresponds to (6.ii) only on condition (sufficiency condition) that $k^o$ configure an actual possibility space. In other words we can say that an expression like (6.ii) succeeds in defining an actual measure only if the sufficiency condition is satisfied.

This condition is nothing but the intentional counter-part of the presupposition upon which the set-theoretic approach to probability is based; in fact (Hayek 2003 §1, Howson and Urbach 2006 §2.a) such an approach presupposes a space of possibilities $\Omega$ (also: a universal set, a class of elementary events) and refers the probability of an event to the members of $\mathcal{F}$, where $\mathcal{F}$ is the field of $\Omega$-subsets.

6.4.2.1. If the measure of an $h$ is actually defined with regards to a basic statute $k^o$, then it is also defined with regards to a statute $k^o&h_1$; in fact if the information $k^o$ satisfies the sufficiency condition for $h$, the information obtained by adding $h_1$ to the previous one configures either the same possibility space ($h_1=\emptyset$) or a more punctual possibility space (that is a possibility space whose alternatives are a subset of the previous and less punctual one). Therefore while any increment of the basic statute does avoid the risk of losing sufficiency, a decrement does not.

For instance, the possibility of assigning a measure to the outcome 5 with reference to the $k^o$ of our mini-roulette entails the possibility of assigning a measure to the same outcome 5 with reference to the aforementioned
statute $k^*\&h_1$ telling us, say, that even outcomes are precluded, but it does not entail the same possibility if we forget the exact numbers of outcomes in such a mini-roulette.

6.4.3. Normally, increasing a quantity is increasing its measure. Therefore, since here we are dealing with pieces of information and the quantity of information $h_1 \& h_2$ is equal ($h_2 = \emptyset$) or greater than $h_1$, we might be induced to think that $\mu(h_1 \& h_2)$ is equal or greater than $\mu(h_1)$. On the contrary we shall see even formally that $\mu(h_1 \& h_2)$ is equal ($h_2 = \emptyset$) or less than $\mu(h_1)$. This conclusion is confirmed by diagrammatical evidence; in fact the virgin field corresponding to $\mu(h_1 \& h_2)$ is equal or less than the virgin field corresponding to $\mu(h_1)$. Let me insist: $\mu$ measures the residual uncertainty. In this sense calling $\mu$ “measure” is a potentially misleading choice. Yet such a choice is constrained by the current terminology; in fact (§7.12) the above defined notion (owing to the role it plays in the definition of probability) is strictly linked with the notion Waismann, Carnap and followers name “measure”.

6.5. Indeed, once its rudiments have been understood, $\otimes$ is a rather intuitive representation. Nevertheless some comments are suitable about the representation of the two connectives (conjunction and negation) through which we can create new pieces of information.

6.6. The conjunction $h_1 \& h_2$ is represented by a shaded field whose sectors are the ones shaded by $h_1$ or by $h_2$. Yet a fundamental ambiguity arises. For instance, once assumed that $h_1$ precludes the last four sectors and $h_2$ precludes the sectors 1 and 8, is their conjunction represented by Figure 6.3 or by Figure 6.4? In other words: are there many or only one degree of shading?

My answer is that assuming (as in §6.1) the opposition between *known* and *unknown* without considering intermediate cognitive situations is assuming an idempotent logic where the reliability of any acquirement cannot be strengthened by iteration. Therefore Figure 6.4 is the right representation.

In other words. Once we assume that a piece of information $h$ belongs to a statute $k$, no assumption is possible according to which, so to say, $h$ super-belong to $k$.

6.6.1. On these grounds implication is defined through conjunction; in fact

$$ (6.v) \quad (h_1 \otimes h_2) \text{ iff } (h_1 \& h_2 = h_1) $$

is immediately legitimated in $\otimes$.

Until (6.v) is considered as a strictly formal definition, it is a sovereign intervention which as such does not require any legitimation, but as soon as we relate $\otimes$ and *implication* the sovereignty fails. In fact defining formally a symbol that previously belongs to the current lexicon with a universally accepted meaning is a risky procedure, because it might facilitate argumentative abuses, that is undue inferences implicitly based on the universally accepted meaning. Yet, enucleating and emphasizing such a risk is already a good manner of keeping it under control. Moreover, in our case, no undue inferences are derivable, since it seems to me evident that in an idempotent logic if $h_1$ implies $h_2$, then adding $h_2$ to $h_1$ does not enhance the same $h_1$, and vice versa if adding $h_2$ to $h_1$ does not enhance $h_1$, this means that knowing $h_1$ is knowing $h_2$ too, therefore that the same $h_1$ implies $h_2$. For instance, with reference to our mini-roulette, if we know that only odd outcomes are possible (shading of the even sectors), to learn that an outcome is not 4 (shading of such a sector) does not increase what we already know. Therefore (6.v) complies with the usual meaning of “$\otimes$”.

If $h_1 \otimes h_2$, I say that $h_1$ is an expansion of $h_2$ and $h_2$ is a restriction of $h_1$.

6.7. By

$$ |h| $$

I mean the dilemma concerning the pair of opposite pieces of information $h$ and $\neg h$ (§2.9.1). A dilemma is $k$-decidable iff $k$ allows to infer which of its horn is true and consequently which is false. Therefore the decidability can indifferently be referred to any of the two opposite hypotheses.
6.8. The negation ~\(h\) is represented by a shaded field whose sectors are exactly the virgin sectors of \(h\). That is: two opposite pieces of information are represented by complementary shadings. It follows that incoherence is represented by a wholly shaded circle; and actually incoherence is an excess of information (precluding every \(k^o\)-compatible alternative, therefore contradicting the basic previous assumption). By \(h \& \sim h = \bot\) and \(\sim (h \& \sim h) = \emptyset\)

I introduce the abbreviative symbols “\(\bot\)” (incoherence, excess of information) and “\(\emptyset\)” (tautology, null information). Therefore \(\bot\) and \(\emptyset\) are respectively represented by a totally shaded and by a totally virgin circle.

6.9. The maximal coherent information regarding a possibility space is represented by shading all the sectors but one; in fact such a virgin sector, telling us exactly the actual alternative, crowns our way to a complete knowledge. In this sense I call “exhaustive” a statute leaving only one free alternative and “\(k^o\)-exhaustive” a piece of information \(h\) such that \(k^o \& h\) is exhaustive. For instance, in its ‘pitch-and-toss-interpretation’ of Figure 6.1, if we know \((h_1)\) that the second outcome was \(B\), to learn \((h_2)\) that no pair of consecutive tosses gave the same outcome is a \(k^o\&h\)-exhaustive acquirement, since it allows us to infer that the sequence is \(ABA\). Yet the same acquirement is not \(k^o\)-exhaustive, since not to know that the second outcome was \(B\), admits also \(BAB\) as a \(k^o\)-compatible sequence.

6.10. Any propositional connective can be formulated in terms of conjunctions and negations; therefore, once we know how to represent conjunctions and negations, we also know how to represent any propositional connection. For instance, let me consider informally the inclusive (OR), the exclusive (NAND) and the partitive or disequivalent (XOR) disjunction (their formal introduction in § 7.10.1); then Figure 6.5, Figure 6.6 and Figure 6.7 represent respectively the three disjunctions of (6.iii) and of (6.vi) with reference to our mini-roulette. The three figures have been obtained through the strictly graphic procedures dictated by the definition of the disjunction under contingent scrutiny, that is, respectively,
- OR: shading the sectors shaded both in the representation of (6.iii) and of (6.vi),
- NAND: shading the virgin sectors both in the representation of (6.iii) and of (6.vi),
- XOR: shading all the sectors shaded in the two preceding figures.

Yet the same figures can also be obtained through simply argumentative procedures. For instance (Figure 6.5) the inclusive disjunction of (6.iii) and (6.vi) precludes only the eventuality of an odd and >4 outcome, because such an outcome would contradict both 6.iii) and (6.vi), therefore it would falsify the same disjunction. And so on.

6.11. Since in next chapters these considerations will be retaken and deepened, I propose a particularly flexible example. A tract of a rail is partitioned in eight consecutive segments of different colours, and an automatic spring mechanism pulls a slider on the rail; the slider moves any time from the same starting point and its final position is exactly determined by a lubber line, so that the eight alternatives correspond to the eight segments where the slider can stop. Although it would be easy to sketch contexts where the final position depends on many parameters (variable friction, non horizontal altimetry et cetera) I assume that the resistances to the motion are the same in every point of the tract, and thus that the distance is the only parameter (monoparametric context). Of course the final position of the slider depends on the impulse \(I\) transmitted by the spring, and this impulse ranges from an \(I_{\text{min}}\) to an \(I_{\text{max}}\) according to which the slider stops respectively at the beginning or at the end of the tract.
First of all the flexibility of the example allows to see Figure 6.1 and 6.2 as concerning simply two different partitions of the tract (the area of every sector is directly proportional to the length of the segment it represents).

Such a flexibility could also allow the introduction of new distinctions based on further connotations. For instance a distinction based on the hypothesis that every segment of the tract is partitioned in three sub-segments, each of them identified by a nuance of the respective colour would increase the respective possibility space and could be represented in the intuitive way (the partition of each sector in three subsectors). Nevertheless, since an octo-partition of the circle is already sufficient to support adequate examples and since a non-uniform assignation would mean overcharging the exposition with arithmetical complications of no theoretical moment (let me repeat: uniformity too, after all, is an assignation of measures, and its peculiarity does not influence the following discourse), the statute \( k^o \) represented in Figure 6.1 will be privileged. Of course once every sector is considered as an elementary entity, every piece of information referring to further distinctions is of no moment. For instance

the slider stopped in the second half of the sector 3 does not increase

the slider stopped in sector 3

since under \( k^o \) the specification adduced by “second half of” is meaningless when referred to a segment of the tract.

6.12. The above representation presupposes the assignation of a precise measure to the various alternatives. Of course any partition where the virgin (free) field is composed by more than one sector represents an uncertainty (the certainty is represented by a monosectorial virgin field whose only sector represents exactly the actual alternative). Yet a ‘meta-uncertainty’ may concern the same assignation of a measure to some alternative. If we are studying the three tosses sequence of a well balanced coin, Figure 6.1 is undoubtedly the right one; but not all contexts are analytically quantifiable. For instance, what about the possible results of a foot-race with eight competitors? Of course we can assign to each of them a measure inspired by our previous information on his chances, so in general obtaining a non-uniform diagram like, say, Figure 6.2. This notwithstanding any single assignation might suffer a margin of ambiguity (different bookmakers might propose different rates). A situation like this could be represented by fuzzy sectors, that is by replacing the various radii which separate the various sectors with sub-sectors more or less narrow according to the margin of ambiguity in the assignation of the respective measure. Anyway I neglect such a topic because my present task is only to sketch the theoretical frame of a representation, not to enter into details of its possible refinements.

6.13. Let \( k=k^o&k' \) be the statute concerning the possibility space under scrutiny. According to the above explained technique (henceforth I call it “by shading”) we shade the \( k' \)-sectors of a circle partitioned in conformity with \( k^o \). But, since shading a sector is precluding it (that is cancelling it from the possible alternatives), another technique (I call it “by re-partitioning”) is performable: drawing a new and completely virgin circle whose sectors are only the virgin sectors of the previous one. For instance, once translated into the re-partitioning technique, the situation represented in Figure 6.7 leads to a uniformly tetrapartitioned circle (uniformly because the four virgin sectors of Figure 6.7 have the same area and the areal ratios of the virgin sectors, obviously, must be conserved).

The re-partitioning technique, follows from interpreting the shadings of this technique as a radical elimination of the respective alternatives from the possibility space (so entailing a re-distribution of the resulting virgin field). Yet the most general approach concerns acquirements that, far from forbidding the radical elimination of an alternative, increase or decrease its respective assignation (that is the area of the sector representing such an alternative). This topic will be faced in §13.6.2.

6.14. The incidental note of §6.3.1.1 deserves a last remark.

I only know that A and B are among the (how many?) competitors of a foot race, and that their chances are the same. This situation, which cannot be represented through a circle because of its fixed area, can be represented through an open rectangle (where two equal sub-rectangles are part of a rectangle whose total area is unknown).
7.1. Here I formalize the informational approach through a propaedeutic system of logical axioms. The reasons why I speak of a propaedeutic system will be explained in §8.7.1. Obviously the variables “\(h\), “\(h_1\)”, et cetera range over pieces of information.

The axioms are

\begin{align*}
\text{AX1 IDENTITY} & \quad \text{if } h \text{ then } h \\
\text{AX2 REPLACEMENT} & \quad \text{if } h_1 \text{ and if } h_1 = h_2 \text{ then } h_2 \\
\text{AX3 ASSOCIABILITY} & \quad \text{if } h_1 \text{ and if } h_2 \text{ then } h_1 \& h_2 \\
\text{AX4 COMMUTABILITY} & \quad \text{if } h_1 \& h_2 \text{, then } h_2 \& h_1 \\
\text{AX5 RESTRICTION} & \quad \text{if } h_1 \& h_2 \text{, then } h_1 \\
\text{AX6 COHERENCE} & \quad \text{if } h_1 \& \neg h_1 \text{, then } h_2 \\
\text{AX7 COMPLEMENTARINESS} & \quad \text{if } h_1 \& h_2 = h_1 \& \neg h_1 \text{ then } h_1 \& \neg h_2 = h_1 \\
\end{align*}

and could be re-proposed in the well known fractional notation where for instance AX2 becomes

\[ h_1 = h_2 \]

et cetera. I preferred “if then” only for minute typographical convenience.

7.1.1. The above axioms draw an idempotent logic (§1.3). In fact

\begin{align*}
\text{Theor1. If } h_1 \text{ then } h_1 \& h_2 \text{, and if } h_1 \& h_2 \text{, then } h_1.
\end{align*}

Proof, By AX1 and AX3, if \(h_1\) then \(h_1 \& h_2\); by AX5, if \(h_1 \& h_2\), then \(h_1\).

Obviously the theorem of idempotence concerns pieces of information. For instance it states that, if we know that 2<3, we can infer that 2<3 and 2<3; therefore, since reciprocally (AX5) if we know that 2<3 and 2<3, we can infer that 2<3, to know that 2<3 and 2<3 is nothing more and nothing less than to know that 2<3 (Theor4 below).

7.2. Some comments are opportune in order to show that the usual interpretation of symbols satisfies the admissibility criterion.

7.2.1. AX1 is obvious: any piece of information can be inferred from its assumption.

7.2.2. AX2 rules the substitution of identity. Indeed “substitution of identity” is a patent oxymoron, because in the meaning of “substitution” there is a component of diversity quite incompatible with the very meaning of “identity” (scholasticism taught: \(\text{si duo idem faciunt, non est idem}\)); this notwithstanding I respect the current terminology. Anyhow the topic will be better analyzed in Chapter 8.

7.2.2.1. Besides the substitution of identity, current theorizations list Modus Ponens as a further inference rule. Here, by the definition (6.vi), it is a theorem. In fact

\begin{align*}
\text{Theor2. If } h_1 \text{ and if } h_1 = h_1 \& h_2 \text{, then } h_2.
\end{align*}

Proof. By AX2 we get \(h_1 \& h_2\), then, by AX5, \(h_1 \& h_2\).

Of course if “\(h_1\)” and “\(h_2\)” were variables over sentences, \(h_1 = h_1 \& h_2\) would be an absurdity.

7.2.3. The admissibility of AX3 follows immediately from the same meaning of “and”, that is from the same *conjunction* (when referred to pieces of information). The repetition of “if” says just that the two acquirements are singularly considered.

7.2.4. At first sight a superficial objection might suggest some perplexity about the admissibility of AX4, that is about the commutative property of conjunction. For instance

(7.i) Ava took a lover and Ava’s husband abandoned her and
(7.ii) Ava’s husband abandoned her and Ava took a lover
are obtained by commutating the same two atomic statements, yet (7.i) and (7.ii) adduce two different pieces of information, otherwise it would be unexplainable why the respective lawyers are quarrelling about (7.i) and (7.ii).
The reply is immediate: (7.i) and (7.ii) are elliptic formulations suggesting that the consequentiality of the facts corresponds to the consequentiality of the atomic statements. Indeed, making (7.i) explicit leads to (7.iii)

Ava took a lover at \( t_1 \) and Ava’s husband abandoned her at \( t_2 \)

exactly as making (7.ii) explicit leads to
(7.iv) Ava’s husband abandoned her at \( t_1 \) and Ava took a lover at \( t_1 \)

therefore the example, far from confuting AX4, corroborates it (both (7.iii) and (7.iv) are perfectly commutative).

7.2.5. While AX2, AX3 and AX4 do satisfy the admissibility criterion also under the dual interpretation which reads “&” as a symbol of inclusive disjunction, AX5 does not, so rejecting this dual interpretation. On the contrary AX5 legitimates in the most obvious way the usual interpretation of “&”: in fact while a conjunction increases the information adduced by each conjunct, the inclusive disjunction decreases the information adduced by each disjunct.

7.2.6. AX6 is nothing but the formalization of the classical *Ex absurdo quodlibet*.

In other words it states that the conjunction of two opposite pieces of information precludes every possible alternative.

7.2.7. AX7 states that two opposite pieces of information do not intersect. Therefore AX6 and AX7 in conjunction define exactly the complementary import of \( ^*\sim^* \) which finds in \( \oplus \) its immediate visualization: the shaded fields of two opposite pieces of information do not leave any virgin sector and do not overlap.

7.3. I recall the definition
(7.v) \( (h_1 \supset h_2) = (h_1 \& h_2 = h_1) \)

(§6.6.1); I also recall that under (7.v) \( h_1 \) is an expansion of \( h_2 \) and \( h_2 \) is a restriction of \( h_1 \).

7.4. Here I list some theorems involving only conjunctions, that is only AX1, AX2, AX3, AX4 and AX5.

**Theor3** \( h_1 \& h_2 \supset h_1 \)

**Proof.** By AX4 \( h_1 \& h_2 \supseteq h_1 \& h_1 \& h_2 \)

By Theor1 \( h_1 \& h_1 \& h_2 = h_1 \& h_2 \)

By (7.v) \( h_1 \& h_2 \supset h_1 \)

7.4.1. For the sake of concision the proofs of the theorems below are simply sketched or even omitted (when quite elementary).

**Theor4** If \( h_1 \supset h_2 \) and \( h_2 \supset h_1 \), then \( h_1 = h_2 \)

**Proof:** \( h_1 = h_1 \& h_2, h_2 = h_2 \& h_1 \)

By \( h_1 \& h_2 \subseteq h_1 \)

7.4.1.1. Since (Theor1) if \( h \), then \( h \& h \) and if \( h \& h \), then \( h \), it follows from Theor4

**Corollary 4** \( h \& h = h \)

that is the identity between a piece of information and, so to say, its iteration. Yet we must avoid interpreting such a conclusion in an abusive way. A little example. Bob looks at the outcome of this die, and sees a six; yet he is astigmatic, and as such a little doubt remains: a six or a four? He puts on his spectacles and verifies: surely a six. In this sense someone could object that, since the ‘spectacles-assisted’ acquirement strengthens the first one, Corollary 4 is violated.

No violation indeed, since the little doubt concerning Bob’s first acquirement forbids its acceptance as an acquirement. When we accept a piece of information, we assume it as an unobjectionable datum which no further acquirement can strengthen. That is: here we are theorizing an idempotent logic where no intermediate degree of knowledge is admitted between \( ^*\text{known}* \) and \( ^*\text{unknown}* \).

7.4.2. Some other theorems involving only conjunctions.

**Theor5** If \( h_1 = h_2 \), then \( h_1 \supset h_2 \) and \( h_2 \supset h_1 \)

**Proof:** \( h_1 = h_1 \& h_1 = h_1 \& h_2 \)

\( h_2 = h_2 \& h_2 = h_2 \& h_1 \)

**Corollary 5** \( h_1 \supset h_2 \)

7.4.3. Some other theorems involving only conjunctions.

**Theor6** If \( h_1 \supset h_2 \) and \( h_2 \supset h_3 \), then \( h_1 \supset h_3 \)

**Proof:** \( h_1 = h_1 \& h_2, h_2 = h_2 \& h_3 \)

\( h_1 \& h_2 \supset h_3 = h_1 \& h_2 = h_1 \)

(Transitivity)
Theor 7  If $h_1 \supset h_2$ and $h_1 \supset h_3$, then $h_1 \supset (h_2 \& h_3)$

Theor 8  If $h_1 \& h_2 = h_3$, then $h_3 \& h_1 = h_3$

Proof: $h_1 \& h_2 \& h_3 = h_3 \& h_1 \& h_2 = h_1 \& h_3$.

7.4-3. While an implication where

$(h_1 \supset h_2) \& \sim (h_2 \supset h_1)$

is called “simple implication”, an implication where

$(h_1 \supset h_2) \& (h_2 \supset h_1)$

is called “reciprocal implication”. Then Theor 4 and Theor 5 say that identity is nothing but a reciprocal implication.

Though identity is a relation linking every category of referents, Theor 4 and Theor 5 are not so ambitious as to state that any identity can always be interpreted as a reciprocal implication. Since such theorems concern pieces of information, they simply state that, when we are dealing with pieces of information, identity can be conceived as a reciprocal implication and vice versa.

7.5. Some theorems involving also negations, that is also AX 6 and AX 7.

Theor 9. $(h_1 \& \sim h_1) = (h_2 \& \sim h_2)$

Proof. AX6: $(h_1 \& \sim h_1) \supset h_2$. AX6: $(h_1 \& \sim h_1) \supset \sim h_2$. Theor 7: $(h_1 \& \sim h_1) \supset (h_2 \& \sim h_2)$.

Reciprocally $(h_2 \& \sim h_2) \supset (h_1 \& \sim h_1)$. Ergo for Theor 4 $(h_1 \& \sim h_1) = (h_2 \& \sim h_2)$.

Theor 10 If $(h_1 \& h_2 = h_1)$ then $(h_1 \& h_2 = \perp)$

Proof.... $h_2 \& \sim h_2 = \perp; h_1 \& h_2 \& \sim h_2 = \perp$ by AX6; ergo, if $(h_1 \& h_2 = h_1)$, then by substitution $h_1 \& \sim h_2 = \perp$.

Theor 11 If $(h_1 \& h_2 = \perp)$ then $(h_1 \& \sim h_2 = h_1)$

Proof. AX7 and definition of incoherence.

Theor 12 If $h_1 \supset h_2$ then $\sim h_2 \supset \sim h_1$

(Modus Tollens).

Proof Theor 10: $h_1 \& h_2 = \perp$. AX4: $\sim h_2 \& h_1 = \perp$. Theor 11: $\sim h_2 \& h_1 = h_2$.

Theor 13 $h \& \sim h = h$

Proof By definition $h \& \sim h = \perp$. Theor 11: $h \& \sim h = h$.

Theor 14 $h \& \sim h = \sim h$

Proof $h = \sim h$. Theor 13: $h \supset \sim h$; Modus Tollens $\sim h \supset \sim h$; $\sim h \supset h$.

Theor 15 $h = \sim h$

Theor 16 If $h_1 \supset h_2$ and $h_3 \supset h_2$, then $h_3 \supset h_1$

Corollary 16 If $h_1 \supset h_2$ and $h_3 \supset h_2$, then $h_1 \supset h_3$

7.6. The theorems of the propositional calculus can be easily proved. For instance (Kleene 1974, § 23)

Implication (introduction): $(h_1 \& h_2) \supset h_3 \supset (h_1 \supset (h_2 \& h_3))$

Proof. $(h_1 \& h_2) \& h_3 = h_1 \& h_2 \& h_1 \& (h_2 \& h_3) = \perp$. Theor 11: $h_1 \& (h_2 \& h_3) = h_1$.

Modus tollendo ponens. $(\sim h_1 \& \sim h_2) \& \sim h_1 \supset h_2$.

Proof. $\sim h_1 \& \sim h_2 \& \sim h_1 \& \sim h_2 = (h_1 \& h_2) \& (\sim h_1 \& \sim h_2) = \perp$.

and so on.

7.6.1. The reductio ad absurdum is nothing but a technique derivable from the axioms. In fact, given a statute $k$, if $h$ is such that $k \& h = \perp$, by Theor 11 $k \& \sim h = k$, therefore $\sim h$ is implied by the statute.

7.7. The definition
7.7.1. In spite of the manifest similarities connecting *sum* with *conjunction* and *subtraction* with *ablation*, while in mathematics 

\((n_1 + (n_2)) = (n_3)\) and therefore a subtraction can be conceived as the sum of a negative quantity, in the informational logic, evidently, 

\((h_1 - h_2) = (h_2 & \sim h_1)\) does not hold, since cancelling a piece of information is far from stating its opposite (forgetting that Bob loves Ava is far from believing that Bob does not love Ava). Therefore the ablation cannot be conceived as the conjunction of a negative piece of information.

In our minute practice ablations are mainly involved in situations where an absolutely trustworthy new acquirement incompatible with our previous statute constrains us to correct the same statute, therefore to reject (to ablate) some previously accepted pieces of information.

This topic will be deepened in due course. Here I only emphasize that conjoining pieces of information continues being an increasing operation (Theor3).

7.8. The three axioms for \(\mu\) (§6.4) are

**AX8 POSITIVITY**
If \(\sim (k\&h = \perp)\) then \(\mu_k(h) > 0\)

**AX9 OPPOSITION**
\(\mu_k(h) + \mu_k(\sim h) = \mu_k(k)\)

**AX10 CONDIZIONALIZATION**
\(\mu_k(h\&\sim h) = \mu_{k\&h}(h)\)

(AX10 is so called for it is the father of the Principle of Condizionalization).

Since a measure is a number, the relations among measures are relations among numbers. This entails the presence of mathematical symbols such as “>”, “0”, “+”, and also of other commonplace ones such as “/” for the division et cetera. As far as I know, the theories of probability omit the axiomatization of mathematics, and I follow this procedure.

7.8.1. The symbol “-” occurring below does not mean ablation; it does mean subtraction. This notwithstanding such a symbol is not a homonymy bearer because the context overcomes any ambiguity (where “-” connects numbers, it expresses subtraction, where “-” connects pieces of information, it expresses ablation).

7.8.2. Since \(\mu_k(h)\) is represented in \(\mathfrak{p}\) by the \(k\&h\)-virgin field, the admissibility of the three axioms for \(\mu\) is diagrammatically immediate. For instance AX8 states that if the \(k\&h\)-shaded field is not the whole circle, the respective virgin field is not null. Anyhow the diagrammatic interpretation of some theorems is proposed below.

7.9. Contrary to the already listed theorems, indicated by “Theor”, the \(\mu\)-theorems (that is the theorems depending also on the non-logical axioms AX8, AX9 and AX10 will be indicated by “THEOR” in order to facilitate immediate references. The formal discriminating factor between Theors and THEORS, obviously, is the absence or presence of “\(\mu\)”.

**THEOR1** \(\mu_k(k\&h) = \mu_k(h) = \mu_{k\&h}(k\&h) = \mu_{k\&h}(h)\)
Proof. AX10: \(\mu_k(k\&h) = \mu_{k\&h}(h)\)
and so on.

Of course a derivation like
\(\mu_k(h) = \mu_{k\&h}(h) = \mu_{k\&h}(k\&h) = \mu_k(k\&h) = \mu_k(k)\)
is illegitimate because presupposing that the sufficiency condition (§6.4.2) is satisfied by \(\mu_k(h)\) does not imply that such a condition is satisfied by \(\mu_k(k)\) too (and actually, in general, it is not).

**THEOR2** \(\mu_k(\sim k) = 0\).
Proof. AX9: \(\mu_k(h) + \mu_k(\sim h) = \mu_k(k)\).

**THEOR3** \(\mu_{k\&h}(h)\).

**THEOR4** \(\mu_k(h_1 \& h_2) + \mu_k(h_1 \& \sim h_2) = \mu_k(h_1)\).
Proof AX9 and THEOR1:
\[ \mu_{\delta h}(h_2) + \mu_{\delta h}(-h_2) = \mu_{\delta h}(k & h_1) = \mu_k(h_1). \]

Equivalent formulation: THEOR4'
\[ \mu_k(h_1 & h_2) = \mu_k(h_1) - \mu_{\delta h}(-h_2) \]

THEOR4 shows that, given a \( k \), increasing a hypothesis decreases its \( k \)-measure.

THEOR5 \[ \mu_k(h & ~h) = 0 \]
Proof THEOR4 and Theor15:
\[ \mu_k(h & ~h) = \mu_k(h) + \mu_k(h & ~h) = \mu_k(h) + \mu_k(h) \]

Equivalent formulations: THEOR5'
THEOR5'' \[ \mu_k(\bot) = 0 \]

Diagrammatic interpretation. Since the representation of two opposite pieces of information shades the whole circle, its virgin field, quite independently of the statute, is null.

THEOR6 \[ \mu_k(\emptyset) = \mu_k(k) \]
Proof \( \emptyset = \lnot \bot \). AX9: \[ \mu_k(\emptyset) + \mu_k(\lnot \bot) = \mu_k(k) \]

THEOR7 If \( (\mu_k(h) = 0) \) then \( (k & h = \bot) \)
Proof Modus Tollens on AX8.

THEOR8 If \( (k & h = \bot) \) then \( (\mu_k(h) = 0) \).
Until now \( k = \bot \) has only been a particular case, since THEOR8 shows that an incoherent statute entails a null measure for any hypothesis. In §7.13 this point will be deepened.

THEOR9 If \( (k \supset h) \) then \( (\mu_k(h) = \mu_k(k)) \).
Proof Protasis: \( k & h = k \). Substitution: \( \mu_k(k & h) = \mu_k(k) \). THEOR1: \( \mu_k(k & h) = \mu_k(h) \)

Diagrammatic interpretation: if the shaded field of \( k \) includes the whole shaded field of \( h \), the virgin field of \( k & h \) is the virgin field of \( k \).

THEOR10 \[ 0 \leq \mu_k(h) \leq \mu_k(k) \]
Proof AX8 and THEOR5' as for \( 0 \leq \mu_k(h) \); therefore (AX9) \( \mu_k(h) \leq \mu_k(k) \)

Diagrammatic interpretation: the \( k & h \)-virgin field cannot be greater than the \( k \)-virgin field.

THEOR11 \[ \mu_k(h_1 & h_2) \leq \mu_k(h_2) \]
THEOR12 If \( \mu_k(h_1 & h_2) = \mu_k(h_1) \) then \( \mu_k(k & \lnot (h_1 & \lnot h_2)) = \mu_k(k) \).
Proof If \( \mu_k(h_1 & h_2) = \mu_k(h_1) \) then \( \mu_k(h_1 & \lnot h_2) = 0 \) by THEOR4, ergo \( \mu_k(- (h_1 & \lnot h_2)) = \mu_k(k) \) by AX9.

THEOR12 is the \( \mu \)-correspondent of the Deduction Theorem.

THEOR13 If \( (\mu_k(h) = \mu_k(k)) \) then \( (k \supset h) \)
Proof AX9: \( \mu_k(\lnot h) = 0 \). THEOR7: \( k & \lnot h = \bot \). Theor11, \( k & h = k \).

THEOR14 \[ (\lnot (k \supset h)) \Rightarrow (\mu_k(h) < \mu_k(k)) \]
Proof Modus Tollens on THEOR13, and THEOR9.

Diagrammatic interpretation. If the representation of \( h \) shades some \( k \)-virgin sector, the \( k & h \)-virgin field is less than the \( k \)-virgin one.

7.10. The argument proposed in §6.10 can be re-proposed here. Since (AX9)
\[ (7.\text{vii}) \quad \mu_k(\lnot h) = \mu_k(k) - \mu_k(h) \]
and (THEOR4')
\[ (7.\text{viii}) \quad \mu_k(h_1 & h_2) = \mu_k(h_1) - \mu_{\delta h}(-h_2) \]
give us the measures of negations and conjunctions, and since any propositional connective can be formulated in terms of negations and conjunctions, we can derive from (7.\text{vii}) and (7.\text{viii}) the measure of any piece of information adduced by a proper formula of the propositional calculus. Let me enter into some (useful) detail.

7.10.1. The three disjunctions informally introduced in §6.10 can be formally introduced by
\[ (i) \quad (h_1 \lor h_2 \lor \ldots \lor h_n) = (- (h_1 & \lnot h_2 & \ldots & \lnot h_n)) \]
for the inclusive disjunction OR, by
(h_1 \| h_2 |...| h_n) = ...
= (¬(h_1 \& h_2) \& ¬(h_1 \& h_3) \& ... \& ¬(h_1 \& h_n) \& \& (h_2 \& h_3) \& ... \& (h_2 \& h_n) \& ... \& (h_n \& h_1) \& h_n))
for the exclusive disjunction NAND, and by
(h_1 \lor h_2 \lor... \lor h_n) = ((h_1 \lor h_2 \lor... \lor h_n) \& (h_1 \| h_2 |...| h_n))
for the partitive disjunction XOR. The last disjunction is called “partitive” because an n-uple h_1,..., h_n constitutes a proper partition of the k-compatible eventualities iff the two conditions
(k \& ¬h_1 \& ¬h_2 \&... \& ¬h_n) = ∥
(stating that h_1,..., h_n are inclusively disjoined) and
(¬ (h_i= h_j)) ⇒ (k \& h_i \& h_j= ∥)
(stating that h_1,..., h_n are exclusively disjoined) are satisfied.
Accordingly “∥” could almost be read as a synthesis of “∨” and “|”.

7.10.2. As for the measures of the disjunctions I limit myself to the following theorems.

THEOR15 \[ \mu_k(h_1 \| h_2) = \mu_k(h_1) + \mu_k(h_2) - \mu_k(h_1 \& h_2) \]
Proof By (7.vii) and AX9,
\[ \mu_k(h_1 \| h_2) = \mu_k((¬(h_1 \& h_2)) = \mu_k(k) - \mu_k(¬h_1 \& ¬h_2) \]
\[ = \mu_k(k) - \mu_k(h_1) + \mu_k(¬h_1) + \mu_k(¬h_2) \]
\[ = \mu_k(h_1) + \mu_k(h_2) - \mu_k(h_1 \& h_2) \]

THEOR16 \[ \mu_k(h_1 | h_2) = \mu_k(¬h_1) + \mu_k(h_1 \& h_2) \]
Proof \[ \mu_k(h_1 | h_2) = \mu_k((¬h_1 \& h_2)) = \mu_k(h_1) - \mu_k(h_1 \& h_2) \]. Then THEOR4 and AX9.

THEOR17 \[ \mu_k(h_1 \lor h_2) = \mu_k(h_1 \& h_2) + \mu_k(¬h_1 \& ¬h_2) \]

THEOR18 \[ \mu_k(h_1 \lor h_2 \lor... \lor h_n) = \mu_k(h_1 \& h_2 \&... \& h_n) + \]
\[ + \mu_k(h_2 \& h_1 \&... \& h_n) + ...
+ \mu_k(h_n \& h_1 \& h_2 \&... \& h_{n-1}) \]

THEOR19 If \((k=(h_1 \lor h_2 \lor... \lor h_n))\), then
then \[ \mu_k(h_1 \lor h_2 \lor... \lor h_n) = \mu_k(h_1) + \mu_k(h_2) + ...
+ \mu_k(h_n) \]

(Addictiveness) Proof. If \(h_1,..., h_n\) are a proper partition of \(k\)
\[ h_1 = (¬h_2 \& ¬h_3 \&... \& ¬h_n) \]
\[ h_2 = (¬h_1 \& ¬h_3 \&... \& ¬h_n) \]
and so on, therefore
\[ \mu_k(h_1 \& h_2 \&... \& h_n) = \mu_k(h_1) \]
\[ \mu_k(h_2 \& h_1 \&... \& h_n) + \mu_k(h_2) \]
and so on; the addictiveness follows from THEOR18.

7.10.3. A dilemma is a partitive disjunction between two (opposite) alternatives. Therefore, with reference to a dilemma \((n=2, h_2 = ¬h_1)\)

(7.x) \[ \mu_k(h_1 \lor h_2) = \mu_k(h_1 | h_2) = \mu_k(h_1 \& h_2) = \mu_k(h_1) + \mu_k(h_2) = \mu_k(k) \]
follows from THEOR15, THEOR 16 and THEOR17. Indeed (7.x) can shed light upon some poor uses of the various
disjunctions. An example:

if \( h_1 \lor h_2 \lor \ldots \lor h_n \) and \( h_i \) entails \( \neg h_j \) for all \( i \neq j \), then...

is a habitual expression meaning what can be advantageously meant by

If \( h_1 \lor h_2 \lor \ldots \lor h_n \), then...

since the condition that \( h_i \) entails \( \neg h_j \) for all \( i \neq j \) is exactly the condition that the \( n \) inclusively disjoined pieces
of information are also exclusively disjoined (an example in the formulation of THEOR33 below).

7.11. The definition

(7.xi) \( (h_1 \rightarrow h_2) = (\neg h_1 \lor h_2) \)

introduces the (obviously non-primitive) symbol \( \rightarrow \) for the connections I call "pseudo-hypothetics" (pseudo-
hypothetics will be exhaustively analyzed in Chapter 14, specifically destined to conditionals). And

THEOR20 \( \mu_k(h_1 \rightarrow h_2) = \mu_k(\neg h_1) + \mu_k(h_1 \& h_2) \)

(whose proof follows plainly from (7.xi)) gives us the measure of pseudo-hypothetics.

7.12. In §6.3.3 I affirmed that \( \mu \) is strictly linked to the notion Waismann, Carnap and followers call
"measure". In fact by

(7.xii) \( P(h|k) = \frac{\mu_h(k)}{\mu_k(k)} \)

I define the probability of \( h \) given \( k \). The main difference is that, contrary to them, conceiving the measure as an
absolute or unconditional quantity, in my opinion, is an unsustainable thesis (§15.1.2).

Of course the real number that \( \mu \) assigns to a \( h \) as to a \( k^n \) depends also on the unity we choose for \( \mu \); I will deal
with this marginal aspect in §7.15.

7.13. On the grounds of (7.xii) \( (k=L) \) becomes a strict condition in order to avoid \( \mu_k(k)=0 \), that is a null
denominator. Informally I remark that actually the same notion of probability would be senseless when referred to an
incoherent statute.

Although the main stream of the contemporary orthodoxy starts from a monadic (absolute, unconditional)
probability and (either by a definition or by an axiom) introduces the dyadic (relative, conditional) probability as a ratio
between two monadic ones, in my opinion no monadic probability can exist, since the probability of a hypothesis
depends intrinsically on the statute the same hypothesis is referred to. And the same (7.xii) makes evident such a claim.

Yet I am far from criticizing the distinction between prior and posterior probabilities. Such a distinction is a
correct and essential achievement focusing on the variations determined upon the respective probabilistic values by
increments of information. I am claiming that the same notion of an absolute probability is even more insensate than the
already criticized notion of an absolute measure, because \( P \) would result dyadic even if \( \mu \) were monadic. The trap,
generally speaking, is that while an 'unconditional' probability deals with only one statute (the prior), a 'conditional'
probability deals with two (the prior and the posterior); nevertheless this evident difference must be not mistaken for an
untenable difference of valences. In both cases the function concerns two variables (hypothesis and statute), and only in
the latter must two different values of the second variable be accounted for. Thus a sound formalization must always
deal with a dyadic function \( P(h|k) \) and recognize that there are probability problems concerning single values of such a
function, and probability problems concerning the two values corresponding to the prior and to the posterior statute
(prior and posterior as for the increment of information, obviously). Symbolically: there are probability problems
concerning

(7.xiii) \( P(h|k^n) \)

and probability problems concerning the relation between

\[ P(h|k^n \& k') \]

and (7.xiii).

7.13.1. Moreover, a purely dimensional consideration suggests that to define the conditional probability as a
ratio between two absolute probabilities is a rather arrogant procedure. Its arrogance does not regard the legitimacy of
defining the values of a dyadic function as a ratio between the values of two monadic functions; it does regard the
legitimacy of identifying the thus defined dyadic function with the definiens one. I intend that while a definition like

\[ B(x,y) = A(x)/A(y) \]

is a formally unexceptionable procedure, a definition like

\[ A(x,y) = A(x)/A(y) \]

is at the very least an insidious one. A minute example. If \( A(x) \) is a monadic function assigning to every person \( x \) his/her
wealth (computed, say, in US-dollars), the ratio \( A(x)/A(y) \) defines a new dyadic function whose values are no longer
US-dollars, but pure numbers (for instance: since \( x \) is three times richer than \( y \), the ratio between the wealth of \( x \) and the
wealth of \( y \) is 3); therefore it would not be correct to use the same "\( A \)" to indicate this new function too. Of course the
values of both conditional and unconditional probabilities are pure numbers, but this particularity might be only the
mask of a logical abuse: in fact a probability value and a ratio between two probability values continue being two heterogeneous entities, and this heterogeneity cannot at all mirror the real situation, as “probability” keeps a common meaning both in its ‘unconditional’ and in its ‘conditional’ applications.

7.13.2. Perhaps someone might be tempted to object that the notion of monadic probability is not at all insensate since, for instance, when we roll a just purchased die, 1/6 is the absolute value of each outcome. The reply is easy. This value is not absolute, it actually depends on the cognitive endowment implicitly transmitted by “(purchased) die”, since statistical elements suggest us to interpret the word as “perfectly balanced cube”. In fact, almost in their totality, the dice on sale are perfectly balanced cubes. But if the die has been purchased online at www.Falsaria.com, renowned firm specialized in the construction of deceitful dice, the probability of the various outcomes can be assigned only on the ground of a statute partitioning the space of possibilities in some non-uniform way. So 1/6 is only the conditional probability relative to a statistically privileged statute.

Dissembling the intrinsic relational nature of a quantity by implicit assumptions is an unsound yet frequent habit. For instance I am reading that the distance of Proxima Centauri is 4.2 light-years, but for sure my perfect understanding does not entail that the distance is a monadic notion, it simply means that it is a dyadic notion whose second term (our planet) is tacitly understood. And actually such elliptic ways of speaking are only fully legitimated where the context clearly privileges the second term.

7.14. The above considerations show that a Kolmogorov-style axiomatization is not acceptable; in fact
- it proposes the probability as a primitive notion
- it starts from a monadic absolute probability
- it grounds upon a set-theoretic approach whose exasperated extensionality and homonymy-blindness are quite unfit to account for our very gnosiology, often led by intensional processes.

7.15. As for the unity of measure,

\[(\mu_k)(k)=1\]

is the explicit choice complying with the implicit assumptions of the canonical approaches to the measure function. But I am afraid that (7.xiv) also represents the worst choice: in fact, under it, the same number expresses both \(\mu_k(h)\) and \(P(h|k)\), a coincidence that seems to me an awkward help to possible mistakes between two quite distinct quantities.

Hajek (2003, §1) claims that the non-negativity and normalization axioms here resumed in

\[0 \leq P(h|k) \leq 1\]

are largely matter of convention. I disagree: what is largely matter of convention is the choice of the \(\mu\)-unit, but (7.xii) shows that this choice is irrelevant on the \(P\)-range, which will always respect (7.xv). In fact, while both \(\mu_k(h)<0\) and \(\mu_k(h)\neq\mu_k(k)\) are incoherent values, both \(\mu_k(h)=0\) and \(\mu_k(h)=\mu_k(k)\) are exactly the coherent values accounting for the two border cases (of a \(h\) respectively \(k\)-incompatible and \(k\)-implied).

7.16. The well known probability theorems are derivable by the simple application of (7.xii) to the respective axioms and theorems for \(\mu\). In particular

THEOR21 \[P(h|k) + P(\neg h|k) = 1\]
follows directly from AX9,

THEOR22 \[0 \leq P(h|k) \leq 1\]
follows directly from THEOR9,

THEOR23 If \(h_1 \& h_2 = \bot\) then \(P(h_1 \lor h_2|k) = P(h_1|k) + P(h_2|k)\)
follows directly from THEOR15.

Finally, in order to adequate my formulae to the current ones, let me use “\(e\)” as a new variable ranging over ‘evidences’, that is on acquirements increasing a basic statute \(k^o\):

THEOR24 \[P(h|k^o \& e) = \frac{P(h \& e|k^o)}{P(e|k^o)}\]
Proof. To substitute \(k\) with \(k^o \& e\) in (7.xii), to divide numerator and denominator by \(\mu_k(k^o)\) and to simplify.

These four theorems express in dyadic notation the four axioms upon which the usual theories are normally based (Howson and Urbach 2006, §2a); thence the usual theorems could be considered as already proved. Yet I carried out the task of deriving them again not only because of the new dyadic notation, but also because of the formal compromises affecting some current proofs. For example

AXIOM \[P(t)=1\] if \(t\) is a logical truth
THEOREM \[P(\bot)=0\]
Proof \(\sim \bot\) is a logical truth, hence...

(ibidem, §2b, (6)) seems to me a rather rough argument. No doubt that \(\sim \bot\) is a logical truth, but this means only that a theorization entailing such a conclusion is admissible. A formal proof must start from some axiomatic formula and
must transform it into the theoremic one with only the help of the inference rule(s); as such, maintaining the example, no proof can appeal to the notion of logical truth before its formal definition within the system, and no expression can be assumed as a logical truth before its formal derivation.

7.16.1. Other useful probability theorems are

THEOR25  \[ P(h_1 \& h_2 | k & e) = P(h_1 | k & e)P(h_2 | k & e & h_1) \]

Proof  
\[ P(h_1 \& h_2 | k & e) = \frac{\mu_{k \& e}(h_1 \& h_2)}{\mu_{k \& e}(k \& e)} = \left( \frac{\mu_{k \& e}(h_1 \& h_2)}{\mu_{k \& e}(h_1)} \right) \left( \frac{\mu_{k \& e}(h_1)}{\mu_{k \& e}(k \& e)} \right) = P(h_1 | k & e)P(h_2 | k \& e \& h_1) \]

COROLLARY 25  \[ P(h_1 \& h_2 | k) = P(h_1 | k)P(h_2 | k \& h_1) \]

THEOR26  \[ P(h_1 \lor h_2 | k) = P(h_1 | k) + P(h_2 | k) - P((h_1 \& h_2) | k) \]

Proof  THEOR15 and (7.xii)

THEOR27  If \((k = (h_1 \downarrow h_2 \downarrow \ldots \downarrow h_n))\) then \(P(h_1 \downarrow h_2 \downarrow \ldots \downarrow h_n | k) = P(h_1 | k) + P(h_2 | k) + \ldots + P(h_n | k) = 1\)

Proof  THEOR19 and (7.xii).

THEOR28  If \((\mu_0(h_1) = \mu_0(h_2) < \mu_0(k))\) then \(P(h_1 \mid k \& h_2) > P(h_1 \mid k)\)

Proof  AX6: \(\mu_0(h_1) > 0; P(h_1 \mid k \& h_2) = \frac{\mu_{k \& h_2}(h_1) \& k \& h_2}{\mu_{k \& h_2}(k \& h_2)} = \frac{\mu_0(h_1)(k \& h_2)}{\mu_0(h_2)} > \frac{\mu_0(h_1)(k)}{\mu_0(h_2)}\).

THEOR28 is a milestone along the way to inductive inference, since it states that if a coherent \(h_1\) entails a consequence \(h_2\) not \(k\)-entailed then the acquirement of \(h_2\) increases the probability of \((validates) h_1\) given \(k\).

THEOR29  If \((P(h_1 \& h_2 | k) = 1)\), then \((P(h_1 | k) = 1)\)

Proof  From THEOR26 and THEOR22.

THEOR30  If \((P(h_1 | k) = 0)\), then \((P(h_1 \& h_2 | k) = 0)\)

THEOR31  \((\mu_0(k) = \sum_j \mu_0(h_j)) \Rightarrow (\mu_{k \& e}(k \& e) = \sum_j \mu_{k \& e}(h_j))\)

(a proper partition of \(k\) is also a proper partition of \(k \& e\))

Proof  From THEOR19 \((k = h_1 \downarrow h_2 \downarrow \ldots \downarrow h_n) \Rightarrow (\mu_0(k) = \sum_j \mu_0(h_j))\)

THEOR32  \((k = h_1 \downarrow h_2 \downarrow \ldots \downarrow h_n) \Rightarrow (\sum_j P(h_j | k \& e) = 1)\)

THEOR33  \((k = h_1 \downarrow h_2 \downarrow \ldots \downarrow h_n) \Rightarrow (P(h_j | k \& e) = (P(h_j | k)P(e | k \& h_j)) / (\sum_j P(h_j | k)P(e | k \& h_j)))\)

(THEOR33 is Bayes’s Theorem in its complete formulation).
8.1. The formal treatment of alethics could proceed quite independently of any representation. Yet, since \( \& \) helps both the exposition and the understanding of the matter, I will make a large use of diagrams. Of course such diagrams can be applied to whatever universe of reference where the statute allows a partition of the possibility space, that is a codification of the alternatives. Yet, for the sake of simplicity, I will mainly reason about the example sketched in §6.11 (the slider on the rail). Then a generic \( \& \)-diagram represents the basic statute \( k^0 \) (concerning the eight segments of the tract) by partitioning the circle in eight virgin sectors and the acquirements \( k' \), \( k'' \) et cetera by shading the sectors corresponding to the segments of the tract precluded by such acquirements (for the sake of concision, we can only consider a single acquirement).

Our fundamental problem is representing an hypothesis and assigning an alethic value (a ‘truth value’) to it. I emphasize that *alethic value* must be intended in its widest acception, according to which not only *true* and *false*, but also *probable*, *decidable* et cetera are alethic values. The link among such notions is evident. For instance a piece of information \( h \) is \( k \)-true iff \( P(h|k)=1 \) and is \( k \)-false iff \( P(h|k)=0 \). Therefore the intrinsically relational nature of *probable* is the intrinsically relational nature of *true* et cetera. Without a reference to a statute, all alethic predicates are senseless.

8.1.1. In order not to waste time with analyses of too scarce an interest, incoherent statutes will be neglected (that is, formally: \( \neg (k \supset h \& \neg h) \) is a presupposed condition). I recall §6.2: the existence of a coherent statute does not imply the ontological existence of the universe it describes. We can reason about Polyphemus exactly because the objects of logic are pieces of information quite independently on their eventual fictitiousness. Anyhow the privileged role we must recognize to the actual statute; will be treated in Chapter 13.

8.2. An alethic procedure is essentially an informational collation which, as such, can be analysed in the institutive, in the propositive and in the properly collative stages. The institutive stage consists in the assumption of a statute \( k \) (\( k=k^0 \& k' \)) that is of an arbitrary informational endowment constituting the basic element of the collation. The propositive stage consists in the assumption of a hypothesis \( h \), that is of an arbitrary piece of information constituting the element to collate with the basic element. The collative stage consists in the comparison between \( h \) and \( k \). The various alethic predicates correspond to the different results of this comparison.

8.3. Although both an acquirement and a hypothesis are pieces of information, they play an opposite role in any alethic procedure. Let me spend few informal words about such an opposition. This academic hall is crowded by teachers and students. All of them, after all, are human beings, therefore, till the discourse concerns generically the human beings crowing this hall, one only sort of individual variables is sufficient. But, since the role of teachers and students, as for the examinations in course, is opposite; once the discourse involves examinations, the strictly complete symbolic endowment to reason on the universe constituted by the persons crowding the hall ought to list three sorts of individual variables; for instance \( \"x\" \) ranging over the subset of teachers, \( \"y\" \) ranging over the subset of students and \( \"z\" \) ranging over the whole set. We could even accept to renounce \( \"z\" \) (replacing it by a disjunction) but we could never accept further renounces, since a symbolic endowment listing only one sort of variables would mutilate our same expressive power. Analogously, with reference to pieces of information, as soon as we deal with collations, we need at least two sorts of variables as \( \"k\" \) (for cognitions) and \( \"h\" \) (for hypotheses). Correspondingly in \( \& \) we need two sorts of marks. and I agree that in \( \& \) cognitions are represented by shadings and hypotheses by hatchings. Therefore the elementary representation of an alethic procedure can be performed by two \( \& \)-diagrams. Both of them start from the virgin circle representing \( k^0 \): in the first (institutive) diagram the sectors representing the alternatives precluded by \( k' \) are shaded; in the second (propositive) diagram the sectors representing the alternatives precluded by \( h \) are hatched. The collative stage is realized by comparing the relations between shaded field and hatched field. For the sake of simplicity, once agreed that shadings and hatchings can overlap, the two diagrams can be unified, thus helping the comparison.

8.4. Let \( h \) be a hypothesis concerning a statute \( k \). We say

a) that \( h \) is \( k \)-true (symbolically: \( T_k(h) \)) iff \( h \) does not preclude any \( k \)-free alternative

b) that \( h \) is \( k \)-false (symbolically: \( F_k(h) \)) iff \( \neg h \) is \( k \)-true, that is if \( h \) precludes all the \( k \)-free alternatives

c) that \( h \) is \( k \)-decidable (symbolically: \( D_k(h) \)) iff \( h \) is either \( k \)-true or \( k \)-false

d) that \( h \) is \( k \)-undecidable (symbolically: \( U_k(h) \)) iff \( h \) is neither \( k \)-true nor \( k \)-false
8.4.1. In §10.1 the theme concerning the choice of “undecidable” in order to adduce the above agreed piece of information will be deepened.

8.4.2. The notion of h-exhaustiveness (of h-incompleteness) can be strengthened by agreeing that a statute $k$ is absolutely exhaustive (absolutely incomplete) iff it is $h$-exhaustive ($h$-incomplete) for every $h$ concerning its possibility space.

8.5. The definitions of §8.4 are not affected by any arbitrariness: they are dictated by previous assumptions and by the usual meanings of alethic predicates. In fact, owing to the link between *truth* and *probability* (§8.1: a piece of information $h$ is $k$-true iff $P(h|k)=1$ and is $k$-false iff $P(h|k)=0$), $h$ is $k$-true
implies 
\[\mu_k(h) = \mu_k(k)\]
$k \& h = k$
and
$h$ is $k$-false
implies 
\[\mu_k(h) = 0\]
$k \& h = \bot$

Once expressed in terms of measures the definitions
\[U_k(h) = (0 < \mu_k(h) < \mu_k(k))\]
\[D_k(h) = (\sim(\sim(\mu_k(h) = \mu_k(k)) \& \sim(\mu_k(h) = 0)))\]
\[T_k(h) = (\mu_k(h) = \mu_k(k))\]
\[F_k(h) = (\mu_k(h) = 0)\]
evidence immediately some intuitive alethic relations. For instance
that the opposite of a $k$-undecidable hypothesis is $k$-undecidable, too
that the opposite of a $k$-decidable and $k$-true hypothesis is a $k$-decidable and $k$-false one
that for every coherent statute $T_k(\emptyset)$ and $F_k(\bot)$
et cetera.

Let me insist. Alethics is an intrinsically relational doctrine because the truth (or falsity et cetera) of a hypothesis results from its collation with another information (the statute). And the informational approach evidences the intrinsically relational character of alethic predicates. In this sense *true* (*false* et cetera) is the correct notion by which *true* (*false* et cetera) must always be replaced.

8.6. The representation is immediate. So while the hatched field representing a $k$-true hypothesis must respect every $k$-virgin sector (as $\mu_k(h) = \mu_k(k)$), the hatched field representing a $k$-false hypothesis must involve every $k$-virgin sector and the hatched field representing a $k$-undecidable hypothesis must involve some but not every $k$-virgin sectors. Therefore undecidability entails at least two $k$-virgin sectors, that is (obviously) the non-exhaustiveness of the statute (undecidability follows from some kind of ignorance). The problem of verifying an undecidable hypothesis is the problem of acquiring new cognitions, so that the increased shaded field covers either the hatched one (thus making true the hypothesis under scrutiny) or the non-hatched one (thus making it false).

8.6.1. I do not show in detail that every formal interdependence among the various alethic predicates as, for instance,
\[T_k(h) = F_k(\sim h),\]
\[U_k(h) = (\sim T_k(h) \& \sim F_k(h))\]
\[D_k(h) = (\sim(\sim T_k(h) \& \sim F_k(h)))\]
is adequately and unambiguously represented in $\ominus$.

For instance the absolute exhaustiveness is represented by a diagram where only one sector is virgin; in fact whatever complementary bipartition of the circle is such that exactly one of its two fields falls into the previously shaded one, therefore whatever hypothesis is decidable. Analogously the absolute incompleteness is represented by a completely virgin circle.

8.6.1.1. A pedantry. While in §8.6.1 the decidability is defined through the inclusive disjunction $\sim(\sim T_k(h) \& \sim F_k(h))$
in §8.4 it is explained through a partitive disjunction (either true or false). The inaccuracy is only apparent, because the prejudicial condition of coherence entails 

\[ \neg (T_k(h) \& F_k(h)) \]

that is the exclusive component of the partitive disjunction.

8.6.2. Let me insist. In §8 the various alethic predicates correspond to precise diagrammatic situations whose essential discriminating factor is the ‘topological’ relation between the \( k \)-virgin and the \( h \)-hatched fields. Therefore the possible and reciprocally incompatible results of a collation are

I) the hatched field does not involve the \( k \)-virgin field (that is: the whole hatched field falls into the shaded field)
II) the hatched field involves the whole \( k \)-virgin field (and, eventually, a part of the shaded field)
III) the hatched field involves only a part of the \( k \)-virgin field (and, eventually, a part of the shaded field).

The case I represents a \( k \)-true, the case II represents a \( k \)-false and the case III represents a \( k \)-undecidable hypothesis (of course the case III entails an incomplete statute, because a virgin field constituted by only one sector cannot be partially hatched).

8.6.3. The example I am about to analyse starts from the statute \( k=k^o \& k' \) got by adding the acquirement \( k' \) the slider is in the last quarter of the tract to our basic \( k^o \). Then Figure 8.0

![Figure 8.0](image)

represents \( k \). With reference to it,
- the slider is not in the first quarter of the tract \((h_1 \text{ represented in Figure 8.1})\)
- the slider is in the first half of the tract \((h_2 \text{ represented in Figure 8.2})\)
- the slider is in an odd segment \((h_3 \text{ represented in Figure 8.3})\)

are the three different hypotheses under alethic scrutiny.

Once recalled that the \( k \)-measure of a \( h \) is represented by the area of the \( k \& h \)-virgin field, we can diagrammatically infer that \( T_h(h_1) \), that \( F_h(h_2) \) and that \( U_h(h_3) \). In fact

\[ \mu_h(k \& h_1) = \mu_h(k) \]

(both the virgin sectors of Figure 8.0, that is the sectors 7 and 8, are virgin in Figure 8.1 too)

\[ \mu_h(k \& h_2) = 0 \]

(both the virgin sectors of Figure 8.0 are hatched in Figure 8.2)

\[ 0 < \mu_h(k \& h_3) < \mu_h(k) \]

(only one virgin sectors of Figure 8.0 is hatched in Figure 8.3).

Of course the probabilistic values resulting from the areal ratios, that is \( P(h_1|k)=1, P(h_2|k)=0, P(h_3|k)=1/2 \), correspond to our intuitive suggestions.
8.7. A momentous theme concerns the formulation adopted in §7.1 in order to present a system of axioms. First of all I wish to avoid a possible equivocation; the momentousness does not follow from the fact that those variables, instead of ranging as usual over sentences, range over propositions. In fact, for instance, once recalled that “σ” names the semantic relation and once agreed that “e” is a variable ranging over sentences so that “σe” simply means *the piece of information adduced by e*.

and if σ₁e₁ and if σ₂e₂, then σ₁e₁&σ₂e₂

could replace AX3 et cetera. Keeping propositions as objects of axioms overcomes even the problem concerning the oxymoron between *substitution* and *identity* (§7.2.2) for, of course, since σ₁e₁=σ₂e₂ does not imply e₁=e₂ in

if σ₁e₁ and if σ₂e₂, then σ₁e₁

we deal with two different (therefore non identical) sentences aducing the same (therefore identical) piece of information.

8.7.1. The momentousness of the theme (that is the reason why I spoke of a propaedeutic system of axioms) depends on the non-stringness of such formulations: in fact they are omissive.

Let me be meticulous, looking at the same notion of an axiom from a general informational viewpoint. An axiom assigns an alethic value to a piece of information connected in some way with pieces of information whose alethic value is presupposed. In other words, an axiom is an instrument for inferring. Of course, since alethic values depend on the statutes of reference, any inference too depends on the statutes of reference. Actually it is easy to propose inferences concerning various alethic values and various statutes. For instance

if h₁ is k₁&k₁-undecidable, then h₁&h₂ cannot be k₁-true

is a correct inference concerning two different hypotheses and two different statutes.

The essential dependence of alethic predicates on the statute of reference is a necessary consequence of their intrinsically relational nature.

A logic for statements concerning different statutes is a very wide matter (few notes in §15 below). Here I only deal with one only statute and with true pieces of information. These agreements allow us to omit both indications, so for instance reducing an explicit (therefore a non-omissive) formulation like

if h₁ is k₁-true and h₂ is k₂-true, then h₁&h₂ is k₁-true

or better like

(8.i) if Tₖ(h₁) and if Tₖ(h₂) then Tₖ(h₁&h₂)

to an implicit formulation like

if h₁ and h₂, then h₁&h₂

that is to the formulation of AX3 proposed in §7.1. Analogously

if Tₖ(h₁) and if Tₖ(h₁’h₂) then Tₖ(h₁’h₂)

is the explicit formulation of AX2. And so on.

Therefore any inference concerning different statutes must make explicit the various statutes of contingent reference (explicit inferences). That is: only inferences concerning one only statute can omit the respective reference (implicit inferences).

8.7.1.1. A pedantry. In order to avoid any autonymy, “&” and “=” might be replaced by their names as “CONG” and “NEG”. Under this convention, for instance,

(8.ii) If Tₖ(h₁) and if Tₖ(h₂) then Tₖ(CONG(h₁,h₂))

is the re-formulation of (8.i) et cetera. Yet, once this passage has been emphasized, I will prefer (8.i) to (8.ii) merely in order not to go too far from the usual symbolizations.

8.8. I call “competence condition” the fundamental rule according to which, in an implicit inference only pieces of information belonging to the same statute can be used. The violation of the competence condition leads to potential absurdities.

8.8.1. An easy example (all new lines are metalinguistic). While the piece of information adduced by

(8.iii) Tegucigalpa is the capital of Guatemala

is kₙₚ_TRUE (belongs to Ava’s present statute) because actually Ava thinks so, and the piece of information adduced by

(8.iv) Tegucigalpa is the capital of Nicaragua

is kₙₜ_TRUE because actually Bob thinks so, evidently the piece of information adduced by the conjunction of (iii) and (iv), that is the piece of information adduced by

Tegucigalpa is the capital of Guatemala and Tegucigalpa is the capital of Nicaragua

is neither kₙₚ_TRUE nor kₙₜ_TRUE (none of them thinks that Guatemala and Nicaragua have the same capital). Yet as soon as we realize that we are reasoning about two different (and incompatible) statutes, we realize that the competence condition compels us to make explicit the respective references. And indeed, since both the pieces of information adduced by

(8.v) Ava thinks that Tegucigalpa is the capital of Guatemala
and by

Bob thinks that Tegucigalpa is the capital of Nicaragua
are speaker-true, AX3 assures us that the piece of information adduced by their conjunction too, that is
Ava thinks that Tegucigalpa is the capital of Guatemala and
Bob thinks that Tegucigalpa is the capital of Nicaragua
is speaker-true.  

Actually, since we know that Tegucigalpa is the capital of Honduras,

(8.vi) Tegucigalpa is the capital of Honduras
states a speaker-true identity. If we use (8.vi) for a substitution in (8.v) we get
(8.vii) Ava thinks that the capital of Honduras is the capital of Guatemala
that is a sentence whose interpretation, in spite of its resemblance to (8.v), is ambiguous. In fact if we read (8.vii) as
(8.viii) the piece of information adduced by “the capital of Honduras
is the capital of Guatemala” belongs to $k_{svearer}$
we face a sentence adducing a false proposition (Ava does not at all think that Honduras and Guatemala have the same capital). On the contrary if we read (8.vii) as
(8.ix) Ava erroneously thinks that the town which is the real
capital of Honduras is the capital of Guatemala
we face a sentence adding a true proposition.

And the opposite alethic values of (8.viii) and (8.ix) are the due consequences of the competence condition. In fact the falsity of (8.viii) (recycling use of a new line, what can be false is a proposition, not a sentence), the falsity of (8.viii), then, follows from the violation of the same condition: as the piece of information adduced by (8.vi) does belong to $k_{svearer}$ but, like (8.iv), does not belong to $k_{believe}$, it cannot be used for a substitution within the scope of “belongs to $k_{svearer}$”. On the contrary the truth of the piece of information adduced by (8.ix) follows from respecting the mentioned condition: in fact such an interpretation refers to $k_{svearer}$, and as both the pieces of information adduced by (8.v) and by (8.vi) belong to such a statute, the identity stated by (8.xi) can be used for a substitution in (8.v) because we are looking at Ava’s beliefs from the speaker’s viewpoint (that is because the statute of reference is $k_{svearer}$).

8.8.2. The chronological dimension is not involved in the example above. Yet, of course, the difference between two statutes may also depend exclusively on a difference between the temporal reference. Here is an example..

Though at $t'$ Ava thought (8.iii), at $t$ she realized her mistake, and consequently replaced (8.iii) with (8.vi), which then belongs to $k_{believe}$ too, thus legitimating the inference of (8.vii); nevertheless (8.vii) continues being false, since Ava does not at all think that Honduras and Guatemala have the same capital. But here too we are dealing with two (incompatible) statutes: $k_{svearer}$ (shortly $k$) and $k_{believe}$ (shortly $k'$). If we choose to reason under $k$ we recover the above analysis, and if we choose to reason under $k'$ we cannot use (8.xi) because under $k'$ Ava no longer thinks that Tegucigalpa is the capital of Guatemala. What we can correctly argue is that, since
at $t'$ Ava thought that Tegucigalpa is the capital of Guatemala
belongs to $k'$ (Ava remembers her previous erroneous belief), the competence condition makes
at $t'$ Ava thought that the capital of Honduras is the capital of Guatemala
a $k'$-legitimate inference; and accordingly we recognize that its conclusion is $k'$-true.

8.8.3. The competence condition rules all pieces of information occurring in an inference, and therefore the implicative relations too. So it would be easy to contradict the Theorem of Transitivity (Chapter 6, Theor6) or even the Modus Ponens by an application as

$$k \models h_1 \text{ and if } h_1 \models h_2 \text{ then } k \models h_2$$
and by supposing that $h_1 \models h_2$ does not hold in $k$. For instance (hyperlinguistic new lines)

Flipper is a dolphin
is true for Ava, but, though

*dolphin* implies *mammal*

is speaker-true, since Ava is unaware that dolphins are mammals,

Flipper is a mammal

is not true for Ava. Yet the necessity of respecting the competence condition makes

(8.x)

If $k \models h_1$ and $k \models (h_1 \models h_2)$ then $k \models h_2$

the correct formulation (which, as such, cannot be the source of the well known puzzling questions about oblique contexts). Anyhow this topic will be scrutinized in Appendix 16.

8.8.4. Even the most obvious statement as

if $h$ then $h$

could be immediately contradicted by referring the two occurrences of $h$ to two locally incompatible statutes.

In this sense contexts where one of the different statutes is the speaker’s one (so calling the statute containing previously supplied or universally known pieces of information) are peculiarly insidious.
8.9. The explicit formulations allow a stricter approach to nothing less than the same Aristotelian Non Contradictio (NC) and Tertium Non Datur (TND) principles.

Their usual symbolization, that is

(8.xi) \( \neg(h \& \neg h) \)

for NC and

(8.xii) \( h \lor \neg h \)

for TND, leads to an impasse. In fact, since by definition

\[ h_1 \lor h_2 \]

is

\[ \neg(\neg h_1 \& \neg h_2) \]

and since (Theor15)

\[ \neg \neg h = h \]

it follows that (8.xi) and (8.xii) are equivalent (reciprocally derivable). Quite a disconcerting conclusion, indeed, at least because qualified scholars (intuitionists, for instance) accept the universal validity of NC but not of TND.

I claim that the root of the impasse is exactly the inadequateness of the quoted formulations. What NC and TND intend to establish is that two opposite pieces of information cannot be both true nor both non-true. Therefore both principles are hyperlinguistic statements whose (hyper-)information results from the attribution of an alethic predicate to some (object-)information. In this sense (8.xi) and (8.xii) are mutilating because the alethic predicates are omitted. And as soon as this undue mutilation is corrected by replacing (8.xi) and (8.xii) respectively with

(8.xiii) \( \neg (T(h) \land T(\neg h)) \)

and with

\[ T(h) \lor T(\neg h) \]

that is with

(8.xiv) \( \neg (T(h) \land T(\neg h)) \)

the impasse vanishes; in fact (8.xiii) and (8.xiv) are not at all equivalent. While NC holds both in bivalence and in trivalence (no coherent logic can accept dilemmas \( |h| \) where both \( h \) and \( \neg h \) are true) TND holds only in bivalence (trivalence rules undecidable dilemmas too, and any undecidable dilemma contradicts (8.xiv)). In other words: the simple coherence of a statute \( k \) is sufficient to exclude that both \( h \) and \( \neg h \) can be \( k \)-true, but only the \( |h| \)-exhaustiveness of a statute \( k \) can assure that at least one of the two opposite hypotheses is \( k \)-true. Thus the reason is explained why TND may fail with reference to incomplete statutes: because the non-incompleteness condition limits its universal validity. And just because reality is absolutely exhaustive (the exhaustive coherence of the world we live in is recognized even by some logicians) TND is valid in the logic of an ideal knower.

8.9.1. The definitions of §8.4 allow the formal derivation of the conclusions above (I remind the reader that, in order not to waste time, we are reasoning under the presupposition of coherence for \( k \)).

As for NC. Let \( T(h) \). By definition: \( k \& h = k \). By Theor9: \( k \& h = \bot \). By definition: \( F(h) \). By definition: \( \neg T(h) \). Therefore NC is derivable without any condition of bivalence.

As for TND. Let \( T(h) \lor T(\neg h) \), that is \( \neg (T(h) \land T(\neg h)) \).

By NC: \( \neg (T(h) \land T(\neg h)) \). Therefore either \( T(h) \) or \( T(\neg h) \). By definition either \( T(h) \) or \( F(h) \).

Therefore \( \neg (T(h) \land T(\neg h)) \) that is \( \neg F(h) \). Ergo TND implies bivalence.

8.10. Also the diagrammatic counter-part of the derivations proposed in §8.9.1 is immediate (the \( k \)-coherence entails a non completely shaded circle, that is the presence of at least one virgin sector).

As for NC. The complement of a hatched field falling into the shaded field cannot fall into the same shaded field quite independently on the number of virgin sectors (therefore NC rules also non-exhaustive statutes, that is it rules both bivalence and trivalence).

As for TND. In order to be sure that at least one of two complementary fields falls into the shaded field, the virgin field cannot be formed by more than one sector. Therefore TND is valid only under exhaustive statutes, that is only in bivalence.

8.11. Two last words (referred directly to ®) about the decision of neglecting incoherent statutes. Let me consider a \( k=\bot \), that is a completely shaded circle, that is a \( \mu_F(h) = 0 \). On the one hand I could claim that every hypothesis is \( k \)-true because any hatching falls into the shaded field (here is the representation of the scholastic ex absurdo quodlibet). On the other hand I could also claim that every hypothesis is \( k \)-false, because no virgin sector survives to shading and hatching. The absurdity of the situation, in my opinion, legitimates only one trustworthy conclusion: that incoherence is too deceiving a topic to be analyzed in a rational way. Therefore I firmly insist in neglecting incoherent statutes.

8.12. A possible objection concerning the re-partitioning technique runs as follows.

Let us call "\( k \)-pregnant" a hypothesis iff it concerns such a possibility space, being neither tautologic nor contradictory. While the technique by shading admits pregnant and true hypotheses, the technique by re-partitioning
does not. For instance, if we reduce Figure 8.0 to a virgin circle bi-partitioned in the sectors 7 and 8, no hatching can interest only shaded sectors, therefore no pregnant hypothesis can be true.

Reply. Under the re-partitioning technique those k\(^2\)-pregnant hypotheses are true whose hatching does not interest the virgin circle. I remind the reader (§6.13) that this technique is less complete (it adduces less information) than the technique by shading, where the precluded alternatives too are represented. In other words. Once we agree that the precluded alternatives are neither represented, we agree not to represent any true piece of information.

8.13. The following TABLE 1

<table>
<thead>
<tr>
<th></th>
<th>(h)</th>
<th>(\neg h)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8.xv)</td>
<td>(T_k)</td>
<td>(F_k)</td>
</tr>
<tr>
<td>(8.xvi)</td>
<td>(F_k)</td>
<td>(T_k)</td>
</tr>
<tr>
<td>(8.xvii)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
</tbody>
</table>

rules the alethics of negation; in TABLE 1, while (8.xv) and (8.xvi) concern bivalence, (8.xvii) concerns trivalence. A tetravalent approach to trivalence (no oxymoron) will be proposed in Chapter 9. Here I presuppose that every piece of information we deal with is sorrately correct.

Diagrammatically it is evident that a trivalent logic cannot concern an absolutely exhaustive statute (if \(k\) leaves a monosectorial virgin field, either such a sector is \(h\)-hatched, and then \(h\) is \(k\)-false, or it is \(h\)-virgin, and then \(h\) is \(k\)-true). Since a trivalent logic can only depend on some lack of information, we could say that bivalence is the logic of the ideal knower (or of a human knower whose statute is exhaustive as for the hypotheses under scrutiny).

I emphasize that the compilation of TABLE 1 does not need any integrative assumption; in fact all the alethic values for \(h\) follow theoremically from the respective alethic values for \(h\).

8.14. The following TABLE 2

<table>
<thead>
<tr>
<th></th>
<th>(h_1)</th>
<th>(h_2)</th>
<th>(h_1&amp;h_2)</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8.xviii)</td>
<td>(T_k)</td>
<td>(T_k)</td>
<td>(T_k)</td>
</tr>
<tr>
<td>(8.xix)</td>
<td>(T_k)</td>
<td>(F_k)</td>
<td>(F_k)</td>
</tr>
<tr>
<td>(8.xx)</td>
<td>(F_k)</td>
<td>(T_k)</td>
<td>(F_k)</td>
</tr>
<tr>
<td>(8.xxxii)</td>
<td>(F_k)</td>
<td>(F_k)</td>
<td>(F_k)</td>
</tr>
<tr>
<td>(8.xxii)</td>
<td>(U_k)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
<tr>
<td>(8.xxvii)</td>
<td>(F_k)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
<tr>
<td>(8.xxvii)</td>
<td>(F_k)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
<tr>
<td>(8.xxv)</td>
<td>(F_k)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
<tr>
<td>(8.xxvii)</td>
<td>(F_k)</td>
<td>(U_k)</td>
<td>(U_k)</td>
</tr>
</tbody>
</table>

establishes the alethics of conjunction in its simplest version. While (8.xxviii), (8.xix), (8.xx) and (8.xxxii) concern bivalence, (8.xxii), (8.xxv), (8.xxv) and (8.xxvii) concern trivalence. Also the compilation of TABLE 2 does not need any integrative assumption, since all the alethic values for \(h_1\&h_2\) follow theoremically from the respective alethic values for \(h_1\) and \(h_2\). Let me show this concisely.

The derivations of (8.xviii), (8.xix), (8.xx) and (8.xxxii) are similar. I sketch the derivation of (8.xix): \(T_k(h_j)\) ergo \(k\&h_1\&h_2\&T_k\). Theor7: \(k\&h_1\&h_2\Rightarrow k\). Theor10 \(k\&h_1\&h_2\Rightarrow 1\). By definition \(F_k(h_1\&h_2)\).

The compatibility of \(h_1\) and \(h_2\) is implicitly assured in (8.xxviii) and in (8.xxxii), since two incompatible pieces of information (owing to the coherence of \(k\)) cannot be both \(k\)-true or both \(k\)-false. And the eventual incompatibility of \(h_1\) and \(h_2\) is of no moment in (8.xxvii) and (8.xxv), since their conjunction would anyway be false.

The derivation of (8.xxii) runs as follows. Since \(k\&h_1\&h_2\Rightarrow k\), \(\neg\neg(k\&h_1\&h_2\Rightarrow k\Rightarrow k\&h_1\&h_2\Rightarrow 1\Rightarrow 1\). Theor7: \(k\&h_1\&h_2\Rightarrow k\). Theor10 \(k\&h_1\&h_2\Rightarrow 1\). By definition \(F_k(h_1\&h_2)\).

The incompatibility between \(h_1\) and \(h_2\) is an eventuality that we must reject because if they were incompatible, \(T_k(h_1)\) would entail \(F_k(h_2)\).

The derivation of (8.xxviii), (8.xxv) and (8.xxvii) is analogous.

On the contrary the derivation of (8.xxvii) and (8.xxvii) requires a more detailed analysis. Since \(U_k(h_1)\), \(h_1\&h_2\) cannot be \(k\)-true because if it were \(k\)-true \(k\&h_1\&h_2\Rightarrow k\), therefore \(k\&h_1\&h_2\Rightarrow k\), contrary to \(U_k(h_1)\). So either \(F_k(h_1\&h_2)\) or \(U_k(h_1\&h_2)\). If \(F_k(h_1\&h_2)\), then \(k\&h_1\&h_2\Rightarrow 1\). Theor7: \(k\&h_1\&h_2\Rightarrow 1\Rightarrow 1\). Then \(F_k(h_1\&h_2)\).

The analysis corresponds to the paradigm offered by \(\oplus\). In fact we have two topologically different diagrams representing an \(h_1\) and an \(h_2\) such that \(U_k(h_1)\) and \(U_k(h_2)\). In one of them the two hatchings, once joined, do not leave any virgin sector, and then \(F_k(h_1\&h_2)\). In the other they do, and then \(U_k(h_1\&h_2)\).
For instance, under the usual \( k^o \) (a rail partitioned in 8 segments et cetera), once the slider is in the first half of the tract
is assumed as \( k' \) and the slider is in the first quarter of the tract
is assumed as the \( k \)-undecidable \( h_1 \), if the \( k \)-undecidable \( h_2 \) is
(8.xxvii*) the slider is in the second quarter of the tract
\( h_1 \& h_2 \) is \( k \)-false, while if the \( k \)-undecidable \( h_2 \) is
(8.xxviii*) the slider is in the segment 2
\( h_1 \& h_2 \) is \( k \)-undecidable.

8.15. Since the alethic value of a \( h \) depends strictly on the statute of reference, in general it is impossible to infer the \( k_1 \)-alethic value of a \( h \) from its \( k_2 \)-alethic value, unless \( k_1 \) and \( k_2 \) are linked by some implicative relation.
With the aim of widening our theoretical perspective I propose the following TABLE 3

<table>
<thead>
<tr>
<th>( h )</th>
<th>( h )</th>
<th>( F_k )</th>
<th>( F_{k'} )</th>
<th>( T_{k})</th>
<th>( T_{k'})</th>
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</thead>
<tbody>
<tr>
<td>(8.xxvii)</td>
<td>( T_{k} )</td>
<td>( T_{k'} )</td>
<td>( F_{k'} )</td>
<td>( F_{k'} )</td>
<td></td>
</tr>
<tr>
<td>(8.xxx)</td>
<td>( U_{k} )</td>
<td>( T_{k})</td>
<td>( T_{k'} )</td>
<td>( U_{k})</td>
<td></td>
</tr>
<tr>
<td>(8.xxi)</td>
<td>( T_{k})</td>
<td>( T_{k})</td>
<td>( T_{k'} )</td>
<td>( U_{k})</td>
<td></td>
</tr>
<tr>
<td>(8.xxii)</td>
<td>( F_{k'} )</td>
<td>( F_{k'} )</td>
<td>( T_{k'} )</td>
<td>( U_{k})</td>
<td></td>
</tr>
<tr>
<td>(8.xxiii)</td>
<td>( U_{k} )</td>
<td>( U_{k'} )</td>
<td></td>
<td></td>
<td></td>
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</tbody>
</table>

where the alethic values of an \( h \) are referred to a basic statute \( k^o \) and to a statute \( k^o \& k' \) obtained by increasing \( k^o \) with a cognition \( k' \). Also the compilation of TABLE 3 does not need any integrative assumption, since et cetera. For instance (8.xxviii) ensues from the Theorem of Transitivity. In fact if \( T_{k}(h) \), then by definition \( k^o \Rightarrow h \) , thence, as \( k^o \& k' \Rightarrow k^o \), \( k^o \& k' \Rightarrow h \), that is \( T_{k\&k}(h) \). Analogously for (8.xxix): if \( F_{k}(h) \), then by definition \( k^o \Rightarrow h \) , thence, as \( k^o \& k' \Rightarrow k^o \), \( k^o \& k' \Rightarrow h \), that is \( F_{k\&k}(h) \).

Both (8.xxviii) and (8.xxix) are immediately evident in 
®. For instance, as for (8.xxviii), if the hatching involves no virgin sector of \( k^o \), it involves no virgin sector of \( k^o \& k' \) whose shaded field is either the same (\( k = \varnothing \)) or greater than \( k^o \).

Analogously, as for (8.xxx) if the \( k^o \)-diagram entails that the non-shaded sectors be partially hatched, the further \( k^o \)-shading is compatible with three different situations et cetera.

Not less immediate is the diagrammatic verification of (8.xxxi) (8.xxxii) and (8.xxxiii). The respective formal proofs are obtainable from the same line, here limited to (8.xxxi). \( T_{k\&k} \& F_k \) is contradictory because, owing to (8.xxix), \( F_k \) implies \( F_{k\&k} \) which implies \( \sim F_{k\&k} \) therefore by Modus Tollens \( T_{k\&k} \) implies \( \sim F_k \).

8.15.1. TABLE 3 leads directly to the Theorem of Conservation. Incrementative acquirements of a statute do not alter the truth or falsity of a hypothesis. This theorem ensues directly from (8.xxviii) and (8.xxix) and constitutes a milestone for human knowledge. In fact it assures that, once an assumed statute leads to a conclusion about the alethic value of a hypothesis around, yet they presuppose a correction of the statute, and any correction presupposes an ablation, therefore, first of all, a decrement.

8.15.2. TABLE 3 concerns a single hypothesis. Its complete extrapolation to the case of two hypotheses would entail a too detailed tabulation. I simply emphasize that the task of compiling TABLE 4 as, for instance,

<table>
<thead>
<tr>
<th>( h_1 )</th>
<th>( h_2 )</th>
<th>( h_1 &amp; h_2 )</th>
</tr>
</thead>
<tbody>
<tr>
<td>(8.xxxiv)</td>
<td>( T_k )</td>
<td>( T_{k'} )</td>
</tr>
</tbody>
</table>

is only a matter of patience. So (8.xxxiv), ruling the alethic value corresponding to the conjunction of a \( k^o \)-true \( h_1 \) with a \( k^o \& k' \)-undecidable \( h_2 \), is derivable as follows. If \( T_{k\&k}(h_1 \& h_2) \), then \( T_{k\&k}(h_2) \) incompatible with the presupposition \( U_{k\&k}(h_2) \), therefore \(~(T_{k\&k}(h_1 \& h_2)) \). If \( F_{k\&k}(h_1 \& h_2) \), since on the basis of (8.xxviii) \( T_{k}(h) \Rightarrow T_{k\&k}(h_1) \), then \( F_{k\&k}(h_2) \), incompatible with the presupposition \( U_{k\&k}(h_2) \). Therefore \(~(F_{k\&k}(h_1 \& h_2)) \). But if \(~(T_{k\&k}(h_1 \& h_2)) \) and \(~(F_{k\&k}(h_1 \& h_2)) \) then by definition \( U_{k\&k}(h_1 \& h_2) \).

8.15.3. Far from being a fault, the complexity of the paradigm is quite favourable evidence. In fact, since each line of the above tables refers to a punctual and distinct situation, any approach leading to a less articulate paradigm would only reveal its insufficiency.

8.16. Two important theorems follow.
**Theorem of Restriction:**

(8.xxxv) Every \(k\)-restriction of a \(k\)-true piece of information is necessarily \(k\)-true

(of course a \(k\)-restriction is a restriction valid in \(k\)), then (8.xxxv) is nothing but (8.x), that is the correct formulation of *Modus Ponens*. The formal proof of (8.xxxv) passes through the Theorem of Transitivity, yet it is immediately derivable from TABLE 2 since \(T_k(h_1)\) and \(T_k(h_2)\) is the only combination of values corresponding to \(T_k(h_1 \& h_2)\).

**Theorem of Expansion:**

(8.xxxvi) Every \(k\)-expansion of a \(k\)-false piece of information is necessarily \(k\)-false

(of course a \(k\)-expansion is an expansion valid in \(k\)); (8.xxxvi) is immediately derivable from TABLE 2 since \(F_k(h_1)\) or \(F_k(h_2)\), are separately sufficient to imply \(F_k(h_1 \& h_2)\).

**Corollary:**

(8.xxxvii) No \(k\)-restriction of a \(k\)-undecidable piece of information can be \(k\)-false

(otherwise we could exhibit a \(k\)-undecidable expansion of a \(k\)-false piece of information, so contradicting (8.xxxvi))

On the basis of (8.xxxv), (8.xxxvi) and (8.xxxvii) we can establish a sort of alethic hierarchy among truth (wherever recessive), undecidableness (truth-dominant but falsity-recessive) and falsity (wherever dominant).

8.17. Summarising. Since the information we can infer from a certain acquirement depends also on the statute incremented by the same acquirement, the statute plays a fundamental role: therefore we must be aware of the necessity to inhibit any confusion among different statutes even where no explicit reference occurs. The puzzles affecting intentional identity contexts are born simply by a confusion among the plurality of statutes they involve (in Appendix to Chapter 16 we shall see that inhibiting any confusion is overcoming such puzzles).

Recognizing a plurality of possible statutes, obviously, is perfectly compatible with the incontestable existence of a privileged one, that is, so to say, the final appeal statute concerning the real world we live in (Chapter 13 is specifically devoted to this topic).

8.18. In Chapter 9 the well known distinction between oppositive and exclusive negation will be analyzed carefully. Here sortally incorrect propositions are neglected, and as such only oppositive negations are considered. Nevertheless an even more fundamental distinction concerning negations needs to be focalized.

Let me reason directly on \(R\) and let me consider two complementary propositive diagrams (hatching on a virgin circle); in the former only the sector 3 is not hatched, in the latter only the sector 3 is hatched. Therefore (8.xxxviii) the slider is in sector 3 and respectively (8.xxxix) the slider is not in sector 3 are the sentences adding the represented propositions \(h\) and \(\neg h\).

Now let me suppose that in the institutive diagram representing the statute of reference \(k\) (shading on a virgin circle) the sector 3 is shaded, so making (8.xxxviii) \(k\)-false and (8.xxxix) \(k\)-true. Of course we can state the result of the collation between hatched and shaded fields through (8.xxx) *the slider is not in sector 3* is \(k\)-true but in the usual practice, particularly where the statute of reference is unmistakable, (8.xxx) is replaced by (8.xxxix), which then becomes an ambiguous message. In fact (I continue availing myself of \(R\)) given the representation of a hypothesis (for instance the diagram representing (8.xxxviii)) the “not” through which we mean its complementary diagram (that is the diagram representing (8.xxxix)), is the same “not” through which we mean the (anti-collative) result of the collation between the hypothesis and the statute. Thus we fall into a projective ambiguity (that is an ambiguity involving the dialinguistic order) because a hyperlinguistic statement as (8.xxx) may be confused with a protolinguisitc statement as (8.xxxix).

The above (§8.9) critical approach to the current formulations of NC and TND is nothing but an application of the just proposed considerations. And actually this topic is strictly connected with the already denounced convention (§1.11.1): *the worst symbolic convention ... is the universal habit according to which affirmation is expressed by omitting the symbol of negation.*

8.19. Just as if \(T_k(h_1)\) and \(T_k(h_1 \supset h_2)\) then \(T_k(h_2)\) symbolizes the Theorem of Restriction,
if \( F_k(h_1) \) and \( T_k(h_1\supset h_2) \) then \( F_k(h_2) \)
symbolizes the Theorem of Expansion.

As for undecidable hypotheses

(8.xxxxi) if \( U_k(h_1) \) and \( T_k(h_1\supset h_2) \) then \( \neg F_k(h_2) \)
(in fact if \( F_k(h_2) \) then since \( T_k(h_1\supset h_2) \) by Theorem of Expansion \( F_k(h_1) \), contrary to \( U_k(h_1) \))

(8.xxxxii) if \( U_k(h_1) \) and \( T_k(h_2\supset h_1) \) then \( \neg T_k(h_2) \)
(in fact if \( T_k(h_2) \) then since \( T_k(h_2\supset h_1) \) by Theorem of Restriction \( T_k(h_1) \), contrary to \( U_k(h_1) \)); therefore (I recall that the identity between two pieces of information is a reciprocal implication, and as such it entails both of them)
if \( U_k(h_1) \) and \( T_k(h_1= h_2) \) then \( U_k(h_2) \)
since both (8.xxxxi) and (8.xxxxii) hold.

So, once assumed “\( \mathbb{K} \)” as a variable on alethic values (*true*, *false*, *(un)decidable*), we can resume the above achievements in

if \( S_k(h_1) \) and \( T_k(h_1= h_2) \) then \( S_k(h_2) \)
which tells us that the substitution of identity is an alethically conservative operation (rule of inference). In other words: contrary to Modus Ponens the substitution of identity does not maintain only the truth, but also the falsity and the (un)decidability.
9.1. In this chapter I sketch a truth-functional tetravalent approach to multivalence. I speak of a sketch because the matter is vast. Subsequent chapters (in particular Chapter 15 on variables and Chapter 16 on indexicality) will better enlighten some here apodictic assumptions. Anyhow, in order to simplify all that can be simplified without adulterating the main discourse, I agree
- to consider only propositions adduced by sentences of the atomic form subject-predicate;
- to presuppose the semantic competence of the interpreter, who then can distinguish between sortally correct and sortally incorrect propositions.

Furthermore, in order not to be charged with my own admission (Introduction: too many quotations? they are simply an awkward attempt to mask frightening cultural gaps) many quotations have been censored.

9.2. The hard difficulties entailed by truth-functional approaches to trivalence induced many authors to abandon them; I claim that overcoming a basic inadequacy affecting the current trivalent logics is sufficient to overcoming such difficulties. First of all, let me distinguish between two families of trivalences, respectively called “cognitive” and “sortal”.

9.2.1. The cognitive trivalence, via Łukasiewicz, ascends to Aristotle himself. It refers its third alethic value to some informational gap; while Kripke (1975, p.87) claims that the third alethic value may be interpreted as “possibility”, Kleene (1974, p.333) claims (more clearly, in my opinion) that it means only the absence of information.

When we speak of bivalence or multivalences, we are speaking of alethic values (of their number); and any alethic value results from a collation. Now ascertaining the cognitive impossibility of concluding a collation with a pro-collation (truth) or with an anti-collation (falsity) is ascertaining the undecidability of the piece of information (proposition) under scrutiny ($\S8.4$). Usually ($\S10.1$) this undecidability depends on the poorness of the statute which, as such, does not allow the verification of the unambiguous proposition to collate; yet it may also depend on some ambiguity of the proposition under collation. In other words, the absence of information which usually affects the institutive stage, may affect the propositive stage. For instance, since we know that Ava is a silent but mentally troubled lady, the undecidability of

\begin{quote}
Ava is quiet
\end{quote}

does not depend on our ignorance about Ava’s personality, but on the ambiguity of *quiet*. Nevertheless both an incompleteness of the statute or a fuzziness (Fine 1975) of the proposition entail a lack of information, therefore they can be associated in a classification focused on the distinction between cognitive and sortal trivalences.

9.2.2. The sortal trivalence refers its third alethic value to propositions vitiated by some semantic improperness. For instance while Martin (1968, p.325) writes there is a distinguishable class of odd sentences whose oddity results from a kind of category-incorrectness or non-fitting of subject and predicate, Thomason (1972, p 209) calls “deviant” the sortally incorrect ‘sentences’ (propositions) and writes: the deviation arises from the application of the predicate to something of the wrong sort. The point is clear.

9.2.3. No reasonable confusion is possible between undecidability and improperness. The former needs more information, therefore it presupposes the sortal correctness, otherwise merely semantic considerations would be sufficient to refuse the proposition under scrutiny. At most we can remark that the criterion of interpretative collaboration (the principle of charity) induces us to privilege a proper reading wherever it is possible.

9.3. The mentioned difficulties ($\S9.2$) are well known. In fact (Thomason 1972, p.229) intuition seems to pull in opposite directions: and the presence of incompatible alethic tables for the same connective, as for instance Kleene’s strong and weak ones (Kleene 1974, p. 334) is the direct consequence of such contradictory pulls.

With an eye toward the examples below I agree that

\begin{align*}
(9.\text{i}) & \text{Ava is married} & \text{Ava is pleasant} \\
(9.\text{ii}) & \text{Ava is quiet} & \text{Ava is unquiet} \\
(9.\text{iii}) & \text{Ava is unmarried} & \text{Ava is unpleasant} \\
(9.\text{iv}) & \text{Ava is demonstrable} & \text{Ava is undemonstrable}
\end{align*}

are proper propositions. Furthermore, for the sake of concision, in the examples below pairs of conjoined propositions are reduced to pairs of conjoined predicates et cetera.
9.4. First of all, I treat a strictly cognitive trivalence (where improper propositions are excluded). While $\text{AT}_1$ (where the current and more compact tabulation is adopted)

<table>
<thead>
<tr>
<th>$h_1 &amp; h_2$</th>
<th>$T$</th>
<th>$U$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$U$</td>
<td>$F$</td>
</tr>
<tr>
<td>$U$</td>
<td>$U$</td>
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<tr>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
<td>$F$</td>
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</tbody>
</table>

is the unproblematic table for the conjunction, $\text{AT}_2$

<table>
<thead>
<tr>
<th>$h$</th>
<th>$\sim h$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$F$</td>
</tr>
<tr>
<td>$U$</td>
<td>$U$</td>
</tr>
<tr>
<td>$F$</td>
<td>$T$</td>
</tr>
</tbody>
</table>

is the unproblematic table for the (opposite) negation (symbolized by “$\sim$”). To avoid misunderstandings, alethic tables, generally speaking, say that if certain basic propositions have certain alethic values (if the collation between certain basic propositions and the statute of reference gives certain results), then the proposition obtained by operating on the previous ones through the logical connection under scrutiny have a certain alethic value.

So for instance the “$F$” occurring in the first column, third row of $\text{AT}_1$ says that where

$k \& h_1 = \bot$

and

$k \& h_2 = k$

then

$k \& h_1 \& h_2 = \bot$.

9.4.1. Both $\text{AT}_1$ and $\text{AT}_2$ are unproblematic in the sense that our intuition does not offer room for any tenable alternative in the assignation of their alethic values. For instance, under (9.iii) and (9.ii),

(9.v) Ava is unmarried and quiet

must be false, though we do not know whether Ava is actually quiet, because to know that she is not unmarried is sufficient to conclude that she is not unmarried-and-quiet.

The same structure of $\text{AT}_1$ validates the already remarked hierarchy ($\S$8.16) among alethic values: in cognitive trivalence the dominance scale for conjunction $F > U > T$ complies with the standard interpretation according to which a conjunction takes the value of the ‘least true’ conjunct.

9.4.2. Of course $\text{AT}_1$ and $\text{AT}_2$ allow the compilation of the alethic table for whatever propositional connective in cognitive trivalence. In particular $\text{AT}_3$

<table>
<thead>
<tr>
<th>$h_1 \vee h_2$</th>
<th>$T$</th>
<th>$U$</th>
<th>$F$</th>
</tr>
</thead>
<tbody>
<tr>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
<td>$T$</td>
</tr>
<tr>
<td>$U$</td>
<td>$T$</td>
<td>$U$</td>
<td>$U$</td>
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<tr>
<td>$F$</td>
<td>$T$</td>
<td>$U$</td>
<td>$F$</td>
</tr>
</tbody>
</table>

is the alethic table for the inclusive disjunction (in cognitive trivalence). A passage to emphasize is that though $\text{AT}_3$ has been obtained by a mechanical (a dull?) application of $\text{AT}_1$ and $\text{AT}_2$ to (9.vi)

(not(not ... and not ...))

(that is to the canonical definition of inclusive disjunction), the same $\text{AT}_3$ can also be obtained through the merely intuitive way telling us that an inclusive disjunction takes the value of its ‘most true’ disjunct. So, under (9.iii) and (9.ii)

(9.vii) Ava is unmarried or quiet

must be undecidable; and really, since, contrary to (9.v), if Ava were quiet (9.vii) would be true but if Ava were unquiet (9.vii) would be false, not knowing whether Ava is quiet or unquiet is not knowing whether (9.vii) is true or false. Exactly because in this context truth prevails over undecidability and undecidability prevails over falsity, we can say that the dominance scale for inclusive disjunction in cognitive trivalence is $T > U > F$.

9.5. A strictly sortal trivalence (where no cognitional gaps are admitted) requires a much more meticulous analysis. The logic of improperness, at least from the social viewpoint, is a risky matter; if we insist on submitting improper propositions to our honourable neighbours, their probable reaction is not to debate our sortal theory, but to assure us we are absolutely right and to disappear forever.
This notwithstanding, until we deal with conjunctions, no problem arises. In fact $AT_4$

\[
\begin{array}{c|ccc}
      \hline
      h_1 \& h_2 & T & F & I \\
      \hline
      T  & T  & F  & I  \\
      F  & F  & F  & I  \\
      I  & I  & I  & I  \\
    \end{array}
\]

complies perfectly with our intuition. For instance, under (9.i) and (9.iv),

(9.viii) Ava is married and demonstrable

is manifestly improper ($T \& I = I$, concisely written) because under (9.iv) no married-and-demonstrable lady can exist.

The only (and quite superficial) perplexity might concern $F \& I = I$. Why not $F \& I = F$? The answer is easy. If Ava is unmarried-and-demonstrable

were false, its (opposite) negation ought to be true; then, as

Ava is not unmarried-and-demonstrable is

(9.ix) Ava is unmarried-and-undemonstrable, or married-and-demonstrable, or married-and-undemonstrable

the same (9.ix) ought to be true: is it? I read (9.ix) as a manifestly improper proposition.

Another argument supporting $F \& I = I$ runs as follows. In general, while a false proposition like for instance

$371293$ is prime

is refused on the grounds of a verification, a sortally incorrect proposition like for instance

$371293$ is vegetarian

is refused only on the grounds of semantic considerations; therefore, since a proposition like

$2599051$ is prime-and-vegetarian

does not need any verification in order to be refused (that is; since the arithmetical characteristics of 2599051 are of no moment) the proposition is improper.

So, as our intuition does not offer room for any tenable alternative in the assignation of alethic values, $AT_4$ is the unproblematic table for conjunction in sortal trivalence.

9.5.1. The ticklish problem concerns the table for negation(s). In order to argue in the ordinary language about this matter we have to fix an informal reading for exclusive negation; thus I propose to read the exclusive negation of "$P_\alpha$" as "$\alpha$ cannot at all be $P$" so making "cannot at all be" a declaration of improperness.

Denying a proposition is rejecting its truth. Therefore, until the denied proposition is proper, denying it is affirming the truth of its opposite; but if the denied proposition is improper, its opposite too is improper, and no improper proposition can be true. So for instance,

(9.x) Ava is not unmarried

and

Ava is married are equivalent, but

(9.xi) Ava is not undemonstrable

and

(9.xii) Ava is demonstrable

are far from being equivalent: in fact while (9.xii) is surely improper, in (9.xi) the exclusive reading of "is not" (that is the reading making it a synonym of "cannot at all be"), makes true the same (9.xi). But just because both in (9.x) and in (9.xi) the denial are performed by a "not", as soon as improper propositions are admitted, either we accept "¬" as an intrinsically ambiguous symbol, or we must introduce a second symbol ("¬" say) for exclusive negations. The decision is obvious: ambiguity is the worst enemy of logic.

Unfortunately there is so faint a consent about what exactly an exclusive negation is, that some authors refuse tout court its right to exist. In this sense establishing the respective alethics might reveal itself a rather questionable task. For instance someone could claim that only improper propositions can be denied exclusively or that the exclusive negation of a proper proposition is equivalent to its opposite negation and so on. These different opinions more that symptoms of a puzzling situation, are choices born just by the mentioned vagueness of *exclusive negation*. Is there some puzzle regarding Ava’s quietness, or simply the necessity to achieve a better (heuristic) definition of *quiet*?

My claim is that in sortal trivalence oppositive and exclusive negations are ruled by $AT_5$

\[
\begin{array}{c|ccc}
      \hline
      \h & \neg \h & \neg \h \\
      \hline
      T  & F  & F  \\
      F  & T  & F  \\
      I  & I  & I  \\
    \end{array}
\]

where the only debatable assignations concern the third column (concisely written, $\neg T = F$ and $\neg F = T$ follow directly from $AT_2$ and $\neg I = I$ is banal).
First of all I emphasize that it is impossible to define “¬” in terms of the primitive “&” and “¬”. The simplest way to ascertain this impossibility is to scrutinize the last rows of AT₄ and AT₅. Since in AT₄ “¬” is dominant and in AT₅ ¬I=I, no combination of conjunctions and oppositive negations can lead to a result different from I; on the contrary in AT₅ ¬I=T. Such an assignation follows directly from the same notion of exclusive negation (it is true that Ava cannot at all be demonstrable). On the contrary ¬T=F and ¬F=F are intuitively supported by the evidence that, as *improper* is exactly the opposite of *proper*, the exclusive negation of a proper proposition is false (not improper); for instance, stating that Ava cannot at all be (un)married is false, not improper.

9.6. The basic idea of a tetravalent approach to trivalence is simple: since proper and improper statements can be freely conjoined (as for instance in (9.viii)) and since both proper and improper statements can be oppositively or exclusively denied, the two proposed theorizations can be unified through a table for conjunction and a table for negations where both U and I occur. The task is unequivocally accomplished on the only grounds of the above tables, that is on the only grounds of the different rank U and I have in the respective hierarchies. While AT₁ tells us that in conjunction U, so to say, is F-recessive, AT₂ tells us that F is I-recessive.

In other words. The tetravalent approach follows from a paradigm according to which a proposition is (9.xiii)
- either P (proper, sortally correct) or I (improper, sortally incorrect)
- whether proper, either D (decidable) or U (undecidable)
- whether decidable (therefore proper), either T (true) or F (false).

Of course (9.xiii) is not the only possible paradigm. In my opinion it is the best one yet, in order not to be involved in superfluous quarrels, I simply claim that it is the reasonable paradigm here followed.

So AT₆

\[
\begin{array}{c|c|c|c|c|c}
\hline
h₁\&h₂ & T & U & F & I \\
\hline
T & T & U & F & I \\
U & U & U & F & I \\
F & F & F & F & I \\
I & I & I & I & I \\
\hline
\end{array}
\]

And AT₇

\[
\begin{array}{c|c|c|c|c|c}
\hline
\sim & \sim & h & T & F & F \\
\hline
T & F & F & F \\
U & U & F \\
F & T & F \\
I & I & T \\
\hline
\end{array}
\]

are the tetravalent alethic tables for conjunction and negations. Let me emphasize that no arbitrary intervention affects the editing of AT₆ and AT₇ (¬U=F, follows from the same consideration leading to ¬T=F and ¬F=F: for instance, quite independently of the undecidability of Ava’s (un)quietness, to state that Ava cannot at all be quiet is false).

9.6.1. Realizing the different rank of U and I is understanding the root of the puzzles affecting the usual approaches to trivalence: if we pretend to identify U and I in one only value (the value expressed by Thomason’s asterisk, say) we introduce an incurable conflict of dominance.

This passage may be even more evident in AT₈

\[
\begin{array}{c|c|c|c|c|c|c}
\hline
h₁\&h₂ & T & U & F & I \\
\hline
T & T & T & T & T \\
U & U & U & U \\
F & T & U & F & F \\
I & I & T & F \\
\hline
\end{array}
\]

that is in the table for inclusive disjunction where (obviously because of its duality with conjunction) the aforementioned I>F>U>T becomes T>U>F>l.

9.6.1.1. Indeed AT₈ has been compiled by a (dull) application of the previous tables to the canonical scheme (9.xiv) ¬(¬...&¬...) yet its values could also be obtained by the intuitive suggestion, evidence showing that such compilation is not affected by any puzzling situation.

9.7. From AT₆ and AT₇ we can derive any other connective. The set of combinatory connections, owing to the presence of two different primitive symbols for negation, is very numerous. For instance the scheme
bears eight different symbolizations ((9.xiv) is nothing but one of them, precisely that where the three “not”s are read as oppositional negations).

The sake of frankness compels me to confess that I have analyzed only a little fragment of the whole set; what I can assure is that, as far as I verified it, the suggestions of our intuition are satisfied by the values resulting from the (dull?) application of AT₆ and AT₇.

In order to show that the very reason why our intuition seems sometimes to pull in opposite directions is nothing but the ambiguity of the problem under scrutiny I dwell on a specific scheme.

9.8. The scheme concerns the propositional connective

\[(9.xv) \text{ if } h' \text{ then } h''\]

that is the connective which in the truth functional approach to conditionals corresponds to

\[(9.xvi) \text{ not} (h' \text{ and not } h'')\]

(I remind the reader that a systematic theorization of conditionals, that is a wider approach to (9.xv), will be proposed in Chapter 14).

The scheme is here applied to some contingent examples.

9.8.1. Under (9.i) and (9.iv)

(9.xvii) \text{ If Ava is married, then she is demonstrable} instances a conditional with a true protasis and an improper apodosis; so

(9.xviii) \text{ not}(Ava is married and not demonstrable)

is its reading in terms of conjunction and negations. Yet three incompatible arguments lead to three different alethic values for (9.xvii), or indifferently for its equivalent (9.xviii).

Argument I. So that a conditional be proper, both its protasis and its apodosis must be proper; but the apodosis of (9.xvii) is improper, therefore the same conditional is improper.

Argument II. Since no person can be demonstrable or undemonstrable, Ava cannot be married and not demonstrable, just what (9.xviii) states. Therefore (9.xvii) is true.

Argument III. Since Ava is married, and no person is demonstrable, (9.xviii) denies the conjunction of two true propositions. Therefore (9.xvii) is false.

9.8.2. Analogously, on the grounds of (9.ii)

(9.xix) \text{ If Ava is married, then she is quiet} that is

(9.xx) \text{ not}(Ava is married and not quiet)

instances a conditional with a true protasis and an undecidable apodosis.

Of course, new conclusions impose.

Argument IV. The apodosis of (9.xix) is undecidable; yet if it were true, the conditional too would be true, and if the apodosis were false, the conditional too would be false. Therefore (9.xix) is undecidable.

Argument V. Since Ava is married and may be unquiet, we cannot state that Ava is not married-and-unquiet. Therefore (9.xix) is false.

Arguments VI. Since we cannot exclude that Ava is quiet, we cannot state that she is married and not quiet. Therefore (9.xix) is true.

9.8.3. Finally, again under (9.i) and (9.iv)

(9.xxi) \text{ If Ava is demonstrable then she is married} that is

(9.xxii) \text{ not}(Ava is demonstrable and unmarried)

instances a conditional with an improper protasis and a true apodosis. The respective conclusions follow.

Argument VII. Surely (9.xxi) cannot be false, since its apodosis is true.

Argument VIII. So that a conditional be proper, both its protasis and its apodosis must be proper (Argument I); but the protasis of (9.xxi) is improper, therefore the same (9.xxi) is improper.

Argument IX. Since no person can be demonstrable, Ava cannot be demonstrable and unmarried, just what (9.xxii) states; therefore (9.xxi) is true.

9.9. Since in (9.xvi) “not” occurs twice, tetrivalence can distinguish among four interpretations of (9.xvi) (that is, so to say, four sorts of conditionals) respectively listed in columns (7), (8), (9) and (10) of AT₉.
Once emphasized that the alethic values occurring in AT₉ have been obtained in the dullest way, that is through a mechanical application of AT₆ and AT₇ to (9.xvi), I comment briefly the nine arguments above.

9.9.1. The assumptions of the first three arguments correspond to the fourth row of AT₉ (true protasis, improper apodosis). Argument I is valid iff the two negations occurring in (9.xvi) are oppositive, since the exclusive negation of an improper proposition is proper. In particular, while the oppositive negation of

\[ \neg (Ava \text{ is married and undemostrable}) \]

is improper, its exclusive negation is true. In fact

(9.xxiii) \[ \sim (\text{married (and (undemonstrable)}) \]

is married-and-demonstrable or unmarried-and-undeemostrable or unmarried-and-demonstrable

that is the (improper) disjunction of three improper conjunctions; on the contrary

(9.xxiv) \[ \neg (\text{married (and (undeemostrable)}) \]

leads to

Ava cannot at all be married-and-undeemostrable

that is to an unquestionable truth (under the agreed reading of “cannot at all be”)

So, just as the “T” occurring in the fourth row of the column (7) corresponds to (9.xxiii), the “T” occurring in the fourth row of the column (8) corresponds to (9.xxiv). Furthermore, as

(9.xxv) \[ h' \& \neg h'' \]

(Ava is married and cannot at all be undeemostrable) is true, both its inclusive and exclusive negations (columns (9) and (10) respectively) are false. Therefore there is no puzzle between the outcomes of the mechanical procedure and the conclusions suggested by an analysis grounded upon the intuitive evidence.

For the sake of concision I abridge the following comments which can be analogously argued.

9.9.2. The arguments from IV to VI (true protasis, undecided apodosis) corresponds to the second row of AT₉. For instance Argument VI corresponds to column (9), and actually if we read exclusively the second “not” of (9.xx), since it is false that Ava is not at all unquiet, the conjunction is false, therefore its negation is true. Analogously Argument IV corresponds to column (7) et cetera.

9.9.3. The last three arguments correspond to the thirteenth row (improper protasis, true apodosis). Argument VII is verified by the evidence that no “F” occurs in the last four columns of such a row.

Argument VIII corresponds to column (7), or indifferently to column (9), since the improperness of \( h' \) makes improper both (9.xxv) and

(9.xxvi) \[ h' \& \neg h'' \]

therefore both their oppositive negations.

Argument IX corresponds to column (8) or indifferently to column (10) since the truth of the apodosis makes false both its oppositive and exclusive negations, so making (9.xxv) and (9.xxvi) equivalent.

9.10. In conclusion, as soon as we free the truth functional approach from abusive assumptions and undue restrictions, tetravalence, besides avoiding any puzzle, opens wider horizons to our intuition.
10.1. In this chapter I face the problem concerning the assignation of the respective measures to the alternatives partitioning a possibility space (assignation problem).

Even once agreed that henceforth sortal trivalence is abandoned, so that the third alethic value depends exclusively on a lack of information about proper propositions, and even once agreed that such a lack of information depends exclusively on the institutive stage (§9.2.1), a momentous ambiguity survives because, so to say, a statute can be affected by at least two kinds of informational lacks entailing different consequences on the undecidability of the hypothesis \( h \) to collate. In fact sometimes \( h \) is probabilistically valuable and sometimes it is not. In other words, while it is reasonable to classify all the dilemmas whose two horns are true and false as (alethically) decidable, to classify as undecidable all the dilemmas which are not decidable entails (at least) a missing distinction between probabilistically valuable and probabilistically unvaluable dilemmas.

Let me enter into details. For the sake of simplicity I will mainly reason through \( \Omega^0 \).

10.2. In \( \Omega \) the assignation problem is the problem of quantifying the areas of the sectors involved by such fields, that is the problem of establishing the number and the position of the radii separating the various sectors. For instance, with reference to the aforementioned slider on the rail, if the assignation is uniform \( (k_1 \text{ assigns the same measure to the eight alternatives}) \) we obtain Figure 10.1 (reproducing Figure 6.1)

![Figure 10.1](image1)

![Figure 10.2](image2)

exactly as we obtain Figure 10.2 (reproducing Figure 6.2) if the measure \( k_2 \text{ assigns to alternatives 1 and 2 is the double of the measure assigned to alternatives 3, 4, 5, 6 and the quadruple of the measure assigned to alternatives 7 and 8.} \)

Anyhow a statute may also concern only a part of the tract. For instance, while the statute \( k_3 \text{ represented in Figure 10.3} \)

![Figure 10.3](image3)

 tells us that the first half of the tract is uniformly tetra-partitioned and that the slider is not in its first quarter, it does not tell us anything about the second half of the same tract. Under this \( k_3 \), while \( h_1 \)

(10.i) the slider is in segment 1 is a decidable (and false) hypothesis, both \( h_2 \)

(10.ii) the slider is in segment 3

and \( h_3 \)

(10.iii) the slider is in segment 5

are undecidable. Yet a momentous difference exists between (10.ii) and (10.iii): in fact we can assign a probabilistic value \( (1/8, \text{obviously}) \) to the former, but not to the latter. The probability (under \( k \)) of a \( h \) is the ratio between the \( k \)-measure of \( h \) and the \( k \)-measure of the same \( k \), therefore it is represented by the ratio between the respective virgin
fields. So the impossibility to draw the second radium delimitating sector 5 (whose positions we do not know) forbids the identification of the respective measure. An even more undecidable hypothesis is (10.iv) the slider is in segment 7 for the same existence of a segment 7 is unknown.

10.3. In this sense a refinement of the classification proposed in §8.4 is needed. In accordance with it, a h-incomplete statute is either h-adequate (if it allows the assignation of $P(k|k)$) or h-defective. Reciprocally a k-undecidable hypothesis h is either k-valuable (for instance (10.ii)) or k-unvaluable (for instance (10.iii)).

A statute is absolutely adequate iff it is h-adequate for every h concerning its possibility space; an absolutely adequate statute is represented by a totally partitioned circle. Reciprocally a circle whose partition is totally unknown represents an absolutely defective (or empty) statute. For instance Figure 10.1 and 10.2 represent two absolutely adequate statutes. Instead the statute $k_1$ represented in Figure 10.3 is $h_1$-exhaustive, $h_2$-adequate, and $h_3$-defective, so showing that the same statute may be exhaustive, adequate and defective in dependence of the hypothesis under scrutiny.

Of course an absolutely exhaustive statute is represented by a circle whose sectors but one are shaded, that is, under the re-partitioning technique, by a circle which is also its only sector.

10.4. For the sake of pedantry I remark that the same notion of h-adequateness could be ulteriorly refined by the distinction opposing strong and weak h-adequate statutes. Let me sketch such an opposition through an extremely simple pair of connected examples.

Example I. Under the following statute $k_1$
a) in an urn $m$ there are two counters ($m=m_1$)
b) one counter is blue and one is red ($m=m_1+1$)
c) the other physical characteristics of the counters (size, material, shape et cetera) are exactly equal
d) the withdrawing mechanism is chromatically impartial (no privileging photoelectric apparatus)

the theoretic probability of withdrawing a blue counter is $P(B|k_1) = 1/2$ and the frequency we can empirically realize is 1/2 too. In order to express this identity we could say that $k_1$ is strongly adequate as for the mentioned hypothesis.

Example II. A third equal counter is inserted (blue? red? We only know that no colour is privileged). Under this new statute $k_2$ the theoretic probability of withdrawing a blue counter is $P(B|k_2) = 2/3(1/2)+1/3(1/2)$ therefore it is again 1/2. Yet the limit value of the frequency we can empirically realize will be either 2/3 (if $m=m_2+1$) or 1/3 (if $m=m_2+2$) surely not 1/2. In order to express this non-identity between theoretic probability and frequency we could say that $k_2$ is weakly adequate as for the mentioned hypothesis II.

I do not dwell on this rather marginal topic. Simply I remark that the reason of the discrepancy is that the Example II refers to a second order probabilistic problem, that is to a context where the probabilistic value is grounded on data which are in their turn grounded on probabilistic data. This means that in ® the weakly adequate (the second order) partition of the circle does not correspond to any composition of the urn; in fact, since we ignore its real composition, so to say, such a second order partition results from a pondered valuation of the two first order partitions (I mean $m=m_{2+1}$) each of them representing a strongly adequate statute (no doubt that if we compose 10³ urns under $k_2$ the number of $m_{2+1}$ tends to $10^3/2$).

Henceforth I will only speak of adequate statutes, leaving out of consideration the distinction between strong and weak adequateness.

10.5. The considerations above do not involve the theme concerning any eventual subjective intervention in the assumption of a statute Here I tackle it.

Once fixed the hypothesis h under scrutiny, the promotion of an h-defective statute to h-adequateness or of an h-adequate statute to h-exhaustiveness is realized by acquiring new pieces of information improving the same statute. Since the procedure is the same, for the sake of concision I will only treat the former case.

Actually in every moment of our life we need to take decisions about undecidable hypotheses; in fact they concern possibility spaces whose complexity is too high to be analytically framed and whose context does not allow the acquirement of sure data able to promote the previously defective statute we are dealing with. Therefore, in order to take such decisions, we must resort to some vague and personal estimates. A classical example is the bookmaker who rates the next match; of course such rates are inversely proportional to his opinion about the respective chances, but evidently any mathematically quantifiable (therefore objective) analysis of the thousand factors influencing these chances (influencing this assignation of measures) is beyond his cognitive possibilities. In this sense I speak of a subjective intervention. Yet the subjectivism I am speaking of, in opposition to the fundamentalist subjectivism in the style of de Finetti, is a critical and integrative subjectivism. It is critical because the subjective intervention is not
arbitrary in the broad sense that everyone is free to assign the measures he prefers; these measures have to comply with a sensible estimate of the situation, and such an estimate must be strictly inferred from the contingent physical context of the phenomenon under scrutiny (bookmakers who were to pay a high rate for a nearly sure winner would be condemned to a prompt extinction). It is integrative because, though critical, the subjective intervention is legitimate exclusively where an insufficient knowledge of the same context makes a personal assignation the only reasonable alternative to a sterile suspension of any judgement (the classical εποχή). Even a scale of subjectivity could be proposed in compliance with the ratio between objective and subjective components in the statute on whose basis the assignation is achieved.

Two comments are opportune.

10.5.1. Under my severely deterministic approach (but are there deterministic approaches which are not severely deterministic?) *probability* and *ideal knower* are almost incompatible notions, because they lead to an apparent contradiction. On the one hand, for an ig whatever hypothesis is either true or false, therefore there is no room for any properly probabilistic approach. On the other hand, for instance, stating that, when we toss a (well balanced) die, the probability of a certain result is 1/6 is stating an exact and unobjectionable truth.

No contradiction: ig knows the next result because he (she?) knows the specific values of the parametric quantities, knows such a specific rototranslatory run of the die et cetera; but if we make reference to a generic toss we leave institutionally out of consideration the specific parameters characterizing any specific toss, therefore we introduce, so to say, an institutional lack of information subjecting even ig. In this sense the range of probabilistic values has an absolutely objective import.

In other words. The value of 1/6 expresses the unobjectionable fact that 1/6 is exactly the value a correctly performed frequency (henceforth, a reliable frequency or even, shortly, a frequency) tends to.

10.5.2. Carnap (and followers) claim that the frequency is a second kind of probability. In my opinion a frequency is simply the empirical way to reach a certain result, exactly as the time necessary to fill this tub by this water-tap can be experimentally ascertained even when the strange shape of the tub or the sobbing flow from the water-tap renders very hard any mathematic computation. Analogously, where both a theoretical assignation and a reliable frequency are realizable, the latter can control the former (either validating or invalidating it), and where no theoretical assignation is possible, the reliable frequency supplies an otherwise unachievable datum.

Of course an invalidating reliable frequency (a discrepancy between empirical and theoretical assignations), once excluded a too hasty conclusion suggested by some strange contingent peculiarity (as for instance the famous 21 consecutive rouge in Montecarlo) is an unobjectionable symptom of some mistake in the theoretical approach, that is of some mistake affecting the objective computation or (inclusive) the subjective intervention; thus the non-arbitrariness of the same subjective intervention is argued.

10.6. My basic claim is that the assignation of a measure to the various alternatives concerning a possibility space (that is a partition of the circle) is dictated by our knowledge of the physical phenomenon the same possibility space refers to. In order to argue about this claim I enter into details of Ω°: that is I analyse the procedure allowing me to assign a respective measure to the eight possible alternatives (from now on “upshot”, symbolically “u”, is the technical term to mean the result of a throwing). Given a certain impulse (a certain value of the parametric quantity $\mathfrak{S}$) and a statute specifying the various resistances to the motion (frictional, gravitational, aerodynamic, magnetic and so on) that the slider meets from point to point (let $\mathfrak{R}$ be their resultant), rational mechanics establishes where the slider will stop (that is the value of the covered distance $L$ such that $\mathfrak{S}$ is the integral of $\mathfrak{R}dL$ from 0 to $L$). In order to avoid superfluous complications here too ($\S6.11$) I assume that $\mathfrak{R}$ is the same in every point of the tract, so that a direct proportionality exists between covered distances and impulses (let me insist: this simplification is not at all a theoretically reductive trick to introduce implicitly a positional equiprobability). Thanks to such an $\mathfrak{R}$-uniformity the octo-partitioned segment AB of Figure 10.4 can be indifferently interpreted as the iconic representation of the tract or as the (non-iconic) representation of $\mathfrak{S}$ that is of the impulsive range $\mathfrak{S}_{\text{max}}$, $\mathfrak{S}_{\text{min}}$ (where $\mathfrak{S}_{\text{min}}$ and $\mathfrak{S}_{\text{max}}$ are the impulses necessary to reach respectively the initial point of section 1 and the final point of section 8).

10.6.1. The probabilistic character of the problem depends on the fact that $\mathfrak{S}$ is a parametric quantity. Of course the links between *parametric quantity* and *casual variable* are many, yet I follow my way without dwelling on a rather secondary theme; therefore I say that $\Omega^{0}$ is a possibility space ruled by the parametric quantity $\mathfrak{S}$ over the range $\Delta \mathfrak{S}$ (only impulses belonging to $\Delta \mathfrak{S}$ will be considered).

Let me recall (ibidem) the flexibility of the example. Contrary to a die, where, say, an epta-partition of the possibility space would be rather difficult, here it would be easy to replace the octo-partition with an epta-partition, for instance by annexing section 2 to section 3 or by re-partitioning the entire tract. The crucial passage is that, since the potential final positions of the slider are ‘infinite’, any finite partition (for instance our octo-partition) entails that physically different upshots must be considered equivalent as for their probabilistic classification.
So, for instance, let $s_{\text{min}}$ and $s_{\text{max}}$ be the minimal and maximal impulses relative to the section 4; I call “parametric differential (relative to the section 4)” the value $\Delta s = s_{\text{max}} - s_{\text{min}}$. Generically but propaedeutically speaking the parametric differential $\Delta(x)$ relative to a certain parameter $X$ and to a certain upshot $u_j$ is given by all the parametric values determining $u_j$. In the next chapter more punctual considerations will be proposed.

10.6.2. To object that both *parameter* and *parametric differential* are marginal notions because we can speak of probability also with reference to single events, seems to me a naivety. In fact the assignation of a probability to a single event is only possible if we insert the single event in a parametric context. For an omniscient subject the probability $p$ that I will be alive in ten years is anyhow a certainty: either negative or (as I am inclined to hope) positive; in fact his omniscience allows him to know the future of that single and incomparable (in the most modest acceptation) individual I am. But the probabilistic result of any non-omniscient approach will depend on the set I am inserted in: the probability $p_j$ relative to my being a seventy-years old Italian without severe diseases is greater than the probability $p_2$ relative to my being a person whose mother and maternal grand-father died at fifty by ictus; et cetera. And even if we claim that $p$ is the pondered average of the various $p_n$, we cannot avoid the conclusion that the probability of a single event passes necessarily through parametric probabilities.

10.7. Coming back to our slider, concluding that the measure of each alternative (therefore the areas of the eight sectors occurring in a $\diamondsuit$-diagram) must be directly proportional to the respective parametric differentials (therefore to the lengths of the respective segments occurring in Figure 10.4) would be a serious mistake. In fact concluding that the probability of a generic upshot $u_j$ is given by the ratio $\Delta_j / \Delta(\mathfrak{S})$ is postulating implicitly the uniform distribution of the impulses over the whole range $\Delta(\mathfrak{S})$ because, evidently, if the rule determining the charge of the spring privileges some values, the corresponding upshots too are privileged. In other words, such a postulation is abusive, since a uniform distribution of the impulses is a possibility, not a necessity.

10.7.1. I call “density function” (symbolically “$\psi(\mathfrak{S})$”) the function establishing how the various $\mathfrak{S}$-values are distributed over $\Delta(\mathfrak{S})$. So the measure assigned to the $j$-alternative is directly proportional to
\begin{equation}
\Delta_j(\mathfrak{S}) \psi_j(\mathfrak{S})
\end{equation}
where $\psi_j(\mathfrak{S})$ is the average value of the density function on the respective differential (strictly, (10.v) ought to be replaced by the integral of the $\psi$–function from $s_{\text{min}}$ to $s_{\text{max}}$).

The coefficient of proportionality between measures and (10.v) depends on the choice of a unit, but it is of no theoretical moment (§7.15); for instance, with reference to Figure 10.5

(\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure105.png}
\caption{Figure 10.5}
\end{figure}\)

(\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure106.png}
\caption{Figure 10.6}
\end{figure}\)

(\begin{figure}[h]
\centering
\includegraphics[width=\textwidth]{figure107.png}
\caption{Figure 10.7}
\end{figure}\)

(where a generic density function is represented in ordinate), the important factor is the ratio between the area of any single mixtilinear rectangle and the area of the whole figure, and such a ratio, manifestly, is not at all influenced by the unit we choose.

10.7.2. Therefore, shortly,
\begin{equation}
P(u_j|k) = \Delta_j(\mathfrak{S}) \psi_j(\mathfrak{S}) / \Sigma(\Delta(\mathfrak{S}) \psi(\mathfrak{S}))
\end{equation}
is the general formula, and
\begin{equation}
P(u_j|k) = \Delta_j(\mathfrak{S}) / \Sigma(\Delta_j(\mathfrak{S}))
\end{equation}
is the particular formula where the uniformity of the density function allows the respective simplification.
10.7.2.1. From a formal viewpoint it must be remarked that the reason why “$k$” does not occur in the second members of (10.vi) and (10.vii) is that the parametric differential and the density function can only be determined through a statute; however, if we prefer a heavier but more explicit formulation “$\Delta j$” and “$\psi_j$” can be replaced by “$\Delta j_k$” and “$\psi_{j,k}$” respectively.

10.8. Two particular cases are represented in Figure 10.6 and 10.7. In the former the density is uniform (therefore the areas are directly proportional to the parametric differentials (to the bases of the rectangles). In the latter the partition in parametric differentials is uniform, therefore the areas are directly proportional to the density (to the average heights of the rectangles). A doubly particular case is represented in Figure 10.8,

where both the differentials and the density are uniform, and consequently the upshots are equiprobable (the various rectangles have the same area).

Equiprobable upshots (rectangles of the same area) are also obtained where (Figure 10.9) both parametric differentials and densities are non-uniform, but an inverse proportionality exists between them, so that, according to (10.v), their product is invariant. These two kinds of equiprobability can be distinguished by calling “uniform” the former and “compensative” the latter.

10.9. The proposed analysis shows that an assignation, besides depending on the parametric partition, depends on the density function $\psi$ too. Then an easy puzzle (whose solution will be exposed in the next chapter) runs as follows. In the usual games of chances only the parametric partition is ascertained; for instance we see that every number has a sector of equal width in the rotating disk of a roulette, but we do not know the distribution of impulses transmitted by the croupier to the same disk and to the rolling ball: under the obvious condition that the roulette is well balanced, how on earth is the equiprobability of the upshots universally accepted? In other words: how on earth can we presuppose the uniformity of a density function we do not know? Particularly because this universally accepted presupposition is not limited to equiprobable assignations. For instance if we roll two dice, the partition is heteroprobable (1/36 for a two or a twelve, 1/18 for a three or an eleven et cetera until 1/6 for a seven), but this heteroprobability respects exactly the combinatorary mathematics and consequently, in the end, the partition of parametric differentials; therefore Figure 10.6 tells us that here too a uniform $\psi$ is postulated.
11.1. In order to epitomize my reasoning, I specify
- that a cognition is a piece of information belonging to the statute of reference $k$,
- that, once the piece of information derived from a subjective intervention is assumed as a part of a statute, it achieves the full status of a cognition (actually, in the current practice, distinguishing between subjective interventions and objective acquirements is often a difficult task)
- that an assignation is $k$-equiprobable or $k$-uniform ($k$-heteroprobable or $k$-non-uniform) iff it does (not) assign the same $k$-probability (the same $k$-measure) to all the $k$-compatible alternatives
- that, under a $k$ entailing a uniform assignation, a further cognition $k'$ privileges a hypothesis $h$ concerning the possibility space under scrutiny iff $P(h|k&k') > P(h|k)$.

Therefore

$$(11.i) \quad \text{in absence of privileging cognitions the assignation must be equiprobable}$$

is a formulation of the Principle of Indifference as currently intended.

In any theory of probability I know, far from being a theorem, such a Principle represents an integrative criterion which ought to overcome the circularity affecting the classical definition of probability. Yet its validity is highly controversial. Indeed there are many contexts where the uniformity is a very unreasonable integrative criterion. A trivial example: under a defective statute telling us only that the heights of a crew vary from 1,55 m to 1,95m, are we induced to a uniform assignation (according to which 1,55 and 1,75 are equiprobable heights) or rather to a bell-shaped one, according to which the average height is more probable than the extreme ones?

11.2. I refuse (11.i). And since

$$(11.ii) \quad \text{every assignation is legitimated only by the presence of confirming cognitions}$$

is an obvious rule,

$$(11.iii) \quad \text{a heteroprobable assignation is legitimated only by the presence of privileging cognitions}$$

is the unquestionable particularization of (11.ii).

At first sight refusing (11.i) and accepting (11.iii) seems a pathetic contradiction. In fact, since

* heteroprobability is the opposite of *equiprobability and *absence* is the opposite of *presence*,

$$(11.iv) \quad \text{heteroprobability } \supset \text{ presence of privileging cognitions}$$

seems to imply

$$(11.v) \quad \text{absence of privileging cognitions } \supset \text{ equiprobability}$$

by *Modus Tollens*. Yet this schematic deduction is misleading. In fact the apodosis of (11.iv) concerns two oppositions: *presence* vs. *absence*, and *privileging* vs. *non-privileging*. So, since the opposition *equiprobable* vs. *heteroprobable* concerns two assignations, and since both of them must be based upon cognitions, the opposition to consider for the correct application of *Modus Tollens* cannot be *known* vs. *unknown* (cannot be *presence* vs. *absence*), but rather *privileging* vs. *non-privileging*. Therefore

$$(11.vi) \quad \text{presence of non-privileging cognitions } \supset \text{ equiprobability}$$

is the right inference by which (11.v) must be replaced.

Aphoristically: since ignorance is not evidence, an ignorance-supported equiprobability is a logical abuse.

11.2.1. In other words. Every assignation is grounded on a statute. If it is adequate, the absence of privileging cognitions implies the presence of non-privileging cognitions (otherwise the statute could not be adequate); therefore in condition of adequateness (11.i) is unobjectionable simply because it is equivalent to (11.vi). But of course if the statute is adequate, the Principle of Indifference is superfluous, as no integrative intervention is needed in order to perform the assignation complying with the same statute.

11.3. This conclusion is immediately evident in ®: while a really well-grounded assignation follows from an adequate statute and is represented by a specific partition of the circle, ignorance is ignorance of the partition, and as such is not represented by a uniform partition but by a defective partition.

In Chapter 12 this matter will be more carefully approached. Here I examine some (misleading) arguments which might be proposed in order to sustain the Principle of Indifference in its current interpretation.

11.4. A first argument pro Principle of Indifference might run as follows. Usually the text of a problem adduces all of the information necessary to make it a well determinate problem, that is all of the information necessary to solve it. So, when some datum is lacking, we are implicitly induced to think that such a lack can be filled by an intuitive additional cognition clearly suggested by the context. And actually in thousands of ordinary applications such an additional cognition follows from some kind of uniformity. For instance
Given a circle $C$ and an inscribed equilateral triangle $T$, what is the probability $p$ that a point of $C$ belong to $T$?

is spontaneously interpreted as a well determinate problem whose solution is the ratio between the areas of $T$ and $C$, that is $p=3\sqrt{3}/4\pi$. But this solution is right only under the presupposition of equiprobability (uniform density) for the points of $C$, therefore only under a tacit appeal to the Principle of Indifference. If the circle were the target of Robin Hood, the probability would be nearly 1, because all his arrows would run into the immediate nearness of the centre. In other words. Since - we presuppose that (11.vii) formulates a determinate problem,
- without a density function the problem is indeterminate,
- no density function is indicated in (11.vii),
we are induced to think that the omitted density function is the uniform one. Why the uniform one? Not only because the uniform density function is the simplest one, but also and mainly because it is the only one which can be tacitly understood (while there is only one uniform density function, there are infinite non-uniform density functions, so if we refuse the simplest choice, it becomes impossible to determine the problem, as it is impossible to divine what of the non-uniform density functions must be assumed in order to fill the omission).

Reply. The argument shows only that in some contexts we appeal to the Principle of Indifference. Nevertheless in other contexts we appeal to other criteria (for instance in the example of the crew we could appeal to a standard bell-shaped distribution) or even (where the context does not suggest any reasonable way out) we accept the indeterminateness of the problem, thus renouncing any assignation. In this sense claiming that (11.i) constitutes a general rule is a logical abuse.

11.5. A second argument pro Principle of Indifference might be based on the example proposed in §7.13.2. Bob has just purchased a die about which he possesses no information; nevertheless there is nothing abusive in the fact that he assigns the same probability to the six possible outcomes.

Reply. His equiprobable assignation is legitimate, but the legitimation follows from statistical cognitions supporting it. In fact, almost in their totality, the dice on sale are well balanced. If Bob were to have purchased the die at www.Falsaria.com, he would have immediately rejected the equiprobable assignation.

11.6. A third and more subtle argument pro Principle of Indifference runs as follows. Thousands of frequencies relative to honestly performed games of chance prove that their outcomes are equiprobable. Yet (§10.8) it cannot be a compensative equiprobability: in fact, since the parametric differentials are uniform (an honest die is cubic and well balanced, the rotating disk of a honest roulette is partitioned in 37 equal sectors et cetera) the density function too, although unknown, must be uniform.

The concise reply is that a radical discrepancy forbids assimilating the context concerning the slider (§6.11) with the contexts concerning a die or a roulette, since only in the former case a uniformity of the parametric differential and the equiprobability of the upshots imply a uniform density function. The detailed reply below directs us towards a momentous theorization of the matter.

11.6.1. First of all, let me summarise the general frame of the analysis. A probabilistic problem concerns factual contexts where at least one quantity can assume different values (the parametric range is the domain of a function whose co-domain is the set of possible upshots). Usually the different values of the parametric quantity are infinite, exactly as the final configurations they cause. For instance, the infinite different impulses to the slider and consequently the infinite different positions where it stops; analogously, the infinite different tosses to a die and consequently its infinite dynamic destiny (that is the complex sequence of rototranslations ending in its final position on the green baize). Yet a classification (a partition) is agreed upon whose ground physically different final configurations are made equal (in this sense I speak of conventional upshots). For instance a conventional upshot of the slider is constituted by all the punctual positions belonging to the same segment of the tract; analogously a conventional upshot of the die is constituted by all the outcomes where the die, quite independently on its specific rototranslative process and on its final position, stops with a certain side up. The parametric differential (relative to a certain conventional partition) is constituted by all the values of the parametric quantity determining a punctual final configuration which belongs to the conventional upshot under scrutiny (§10.6.1). But at this point we have to recognize that the same notion of parametric differential must be refined in order to distinguish a theoretically momentous discrepancy between two situations, both of them respecting the proposed condition. I clarify, and for the sake of concision henceforth I will speak of upshots to mean final configurations belonging to a conventional partition.

Let $s_j$ and $s_k$ be two values leading to a same upshot $u$. I say that $s_j \prec s_k$ is a compact parametric differential iff every $s_j$ such that $s_j \prec s_k \prec s_j$ leads to $u$. But if the number of upshots is less than the number of parametric differentials, this means that there are distinct compact parametric differentials leading to the same upshot. In this sense I will speak of composite parametric differentials to mean the union of all the compact ones leading to the same upshot. That is, symbolically, $\Delta \delta = \Sigma \delta_{j,m} \delta$, where $\Delta \delta$ is the composite differential leading to $u$ and $\delta_{j,m}$ is the generic compact differential leading to $u$. 
Accordingly, I will speak of macropartitions to mean partitions where all parametric differentials are compact (that is where the number of upshots is also the number of parametric differentials) and of micropartitions in the contrary case. The radical discrepancy mentioned in the concise reply of §6 above is that while a rail partitioned in eight segments determines a macropartition, a rolled die or a roulette determines micropartitions. In order to emphasize the theoretical importance of this discrepancy I insert the slider in a new context.

11.6.2. The possible upshots are always 8, each of them identified by a specific colour (so for instance $u_1$ is red, $u_2$ is yellow and so forth until $u_8$, blue). Yet the tract of the rail is partitioned in eight million equal sections (the parametrical range is uniformly partitioned in eight million compact microdifferentials $\delta \mathcal{S}$) so that the sections 1, 9, 17, ...7999993 are red, the sections 2, 10, 18, ...7999994 are yellow, and so forth until the sections 8, 16, 24, ...8000000 which are blue. The crucial point is that, under this partition, it is not necessary to suppose a uniform density function in order to get a uniform assignation. In fact, for instance, even a density function like the one represented in Figure 10.5 compensates statistically (I would dare to write “by entropy”) the one million occurrences of any colour. Let me insist: the same (and non-uniform) density function that under a macropartition would entail a non-uniform assignation, under a micropartition entails a uniform assignation.

11.6.2.1. Objection. There are peculiar density functions which present different average values for the various upshots. For instance if we miniaturize Figure 10.5 so that its new range is $\mathcal{S} = \mathcal{S}/10^2$, the density function reproducing one million times that same function assigns to the various upshots the same values of Figure 10.5.

Reply. These ad hoc density functions constitute only an infinitesimal part of the possible ones, therefore, again for statistic reasons, the assumption of a common average density is highly justified. Furthermore it is totally implausible to claim that a croupier is able to realize manually one of these ad hoc functions. Both considerations represent a strong cognition supporting the assumption of equiprobability, which therefore is perfectly compatible with a non-uniform density function.

Comment to the reply. Even supposing that the lengths of the eight million sections, far from being equal, are casually chosen, does not influence the probabilistic conclusions, since here too the average length is anyway the same. In order to exhibit a context where a micropartition leads to a non-uniform assignation we must assure a peculiar length to all the microsections of the same color: but this is so anti-entropic a case (obviously the more micro is the micropartition, the more anti-entropic is the anti(entropic case) that our old world continues coherently running under the opposite hypothesis.

11.7. I epitomise the topic through a schematic paradigm enriched by some remarks. Once agreed that $n$ is the number of converging compact differentials (for the sake of simplicity I assume that every composite differential results from the same number of compact differentials),
- that $\Delta, S = \sum_{\delta \mathcal{S}, \mathcal{S}} \delta \mathcal{S}$
- that $\mu = \sum_{\delta \mathcal{S}, \mathcal{S}} \nu / \mu / n$
- (the composite differential relative to a generic upshot is the sum of the respective compact differentials),
- (the density relative to the composite differential is the average of the respective densities relative to the correspondent $n$ compact differentials),
the general formula is

$$\mu(u_j) = c(\psi, \Delta, \mathcal{S})$$

(11.viii) (that is: the measure of an upshot is directly proportional to the product of its composite differential and the correspondent average density).

Some particular formulas are obtainable by simplifying (11.viii) in accordance with some particular contexts.

11.7.1. Under a non-uniform macropartition and an uniform density function (that is, concisely, under $n=1$,
$$\sim(\Delta, \mathcal{S} = \Delta, \mathcal{S}), \psi = \psi = \psi$$
(11.viii) becomes

$$\mu(u_j) = c^*(\Delta, \mathcal{S})$$

where of cours $c^*$ is $c_2$ and any $\Delta, \mathcal{S}$ is a compact differential. The upshots are heteroprobable in accordance with $\Delta, \mathcal{S}$. It is the situation represented in Figure 10.6.

11.7.1.1. Just as (11.ix) is a particular case of (11.viii),

$$\mu(u_j) = c^{**}$$

(11.x)

is the particular case of (11.ix) where, besides the density, the macropartition too is uniform ($\Delta, \mathcal{S} = \Delta, \mathcal{S} = \Delta, \mathcal{S}$ and $c^{**} = c^* \Delta, \mathcal{S}$). The upshots are equiprobable. It is the situation represented in Figure 10.8.

11.7.2. Under a non-uniform macropartition and a non-uniform density function (that is, concisely, under $n=1$,
$$\sim(\Delta, \mathcal{S} = \Delta, \mathcal{S}), \sim(\psi = \psi)$$
the only simplification of (11.viii) is that no sum and no average are needed. The probability of the upshots is directly proportional to $\psi \Delta, \mathcal{S}$. Then, in general, the upshots will be heteroprobable (it is the situation represented in 10.5); yet a compensative equiprobability follows from an extremely ad hoc and non-uniform density
function counterbalancing the non-uniformity of the parametric differentials, so assuring the invariance of (11.viii). It is the situation represented in Figure 10.9, where all the mixtilinear rectangles have the same area.

11.7.2.1. Under the particular case of an uniform macropartition, (11.viii) becomes

\[ \mu(\psi) = c^{***} \psi_j \]

where of course \( c^{***} = c \Delta \phi \). The upshots are heteroprobable in accordance with the density function. It is the situation represented in Figure 10.7.

11.7.3. Uniform micropartition. The upshots are analytically equiprobable under a uniform \( \psi \) and statistically equiprobable under a non-uniform \( \psi \) because, statistically, also in the latter case \( \psi = \psi \nu \), that is because, statistically, also in the latter case \( \Sigma_{\psi(1,\nu)} \psi = \Sigma_{\psi(1,\nu)} \psi \nu \).

11.7.4. Contrary to uniform micropartitions, which are always non-privileging, both privileging and non-privileging non-uniform micropartitions do exist. I make my point clearer with reference to our canonical example.

11.7.4.1. A non-uniform and privileging micropartition results from a miniaturization similar to the one proposed in §11.6.2.1, provided that here we are under condition of non-uniformity. Since the ratio \( \delta_j / \delta_i \) is always the same, it reproduces itself in the ratio \( \Delta_j / \Delta_i \); and since the micropartition entails \( \Sigma_{\psi(1,\nu)} \psi = \Sigma_{\psi(1,\nu)} \psi \nu \), we are in a situation analogous to §11.7.1 (non uniform macropartitions with uniform density). A non-uniform and privileging micropartition could also be easily realized by a deformed die, or by a cyclic context (for instance by a roulette where the rotating disk is fixed, its various spaces are not the same for every number, and the parametric range of the rolled ball is wide).

Only for the sake of completeness I evoke the theoretical possibility of a compensative equiprobability.

11.7.4.2. A non-uniform and non-privileging micropartition is realized where the microdifferentials are different, but casually determined (for instance: the rail is partitioned in eight million microsegments following the chromatic order, the possible lengths are eight, but the eight million assignations of a length to any microsegment are casual). In such a context, though \( \sim (\delta_j = \delta_i) \), \( \Delta_j = \Delta_i \).

11.8. At this point the solution of the puzzle proposed in §10.9 ought to be clear: owing to the micropartition of impulses transmitted by the croupier to the rotating disk and to the rolling ball, the density function is statistically uniform, then the upshots are equiprobable. Yet I hope that some more detailed reflections will be welcomed, also because the same notion of probability, historically, was born by questions about games of chance.

First of all I focus on the notions of compact and composite parametric differentials making reference to the simplest game of chance, that is to a coin rolled by a mechanized apparatus (the extrapolation to others games of chance is immediate). The range of the possible impulses transmitted to the coin is wide, but it can be partitioned in many compact microdifferentials, each of them delimited (in accordance with the definition) by those impulsive microvariations which should determine the opposite upshot (that is, roughly, those impulsive microvariations which would modify the number of roto-translations of the coin) and the union of all the microdifferentials converging on a single upshot is its composite impulsive differential.

A serious game of chance must assure a basic equiprobability. Yet a compensative equiprobability is practically unrealizable (for instance, in order to realize it we should compensate a deceptive coin through an impulsive program privileging at the same ratio the unfavoured side). Therefore the equiprobability is uniform. The paradigm shows that a uniform equiprobability can be realized a) either by the context of §11.7.1.1 (uniform macropartition and uniform density).

b) or by the context of §11.7.3, that is

b1) uniform micropartition and uniform density
b2) uniform micropartition and non-uniform density.

Then, since no usual game of chance can assure a uniform density, the equiprobability of the upshots follows from b2). Why? Because b2) is the simplest and best context to avoid the previous programmability of certain upshots (trivially: to avoid tricks). In fact the various trials are not realized through mechanized apparatus, but through manual interventions, and within a macropartition a short training would be sufficient to instruct the croupier in obtaining certain upshots (for instance, to instruct him in opportunely impulsion the slider on the basis of the actual stakes). In other words. The uniformity of the parametric differentials is usually assured by the fact that a normal coin is well balanced, or that a normal die is a cube whose geometrical centre is also its barycentre, or that the various numbers in the disk of a roulette have the same room, or that the sections of our rail have the same length and resistance, et cetera). But under a macropartition of the parametric range, the equiprobability can be assured only by a uniform density, and to alter cunningly this condition is a rather easy task; therefore a) is rejected.

As for b), so to say, the more micro the partition, the more difficult it is to alter cunningly the density function: in fact - the microdifferentials render more difficult to perform a pre-programmed value belonging to one of them,
- the uniformity of the density function is not a condition for the equiprobability of the upshots.

In other words; the same empirical structure of the games of chance has been specifically studied in order to render practically impossible any trick (there is no human croupier able to transmit to the rotating disk of a well balanced roulette and to the rolled ball two impulses leading to a specific upshot). Of course there are other possibilities to cheat (for instance a hidden magnet etcetera), but while these eventualities concern a documentable modification of the context, a diabolically able croupier would not modify it. In this sense I say that in a micropartition the upshots are not pre-programmable.

Indeed the distinction *micro* vs. *macro* is not focused on the pre-programmability of the parametric values, but on the number of distinct compact differentials converging on the same upshot (that is: on the ratio 1/n). The criterion of the pre-programmability is important with regards to the possibility to cheat, but nothing more.

11.8.1. Of course the mentioned statistical reasons weaken with the decreasing of n; for instance if the segments of the tract were only 16 (n=2) the uniformity of the average density values would be a quite debatable conclusion. Yet this is only a theoretical specification; in the actual game of chances n is extremely high and consequently the statistical reasons do not give any pretext to dissent.

11.9. The best evidence showing that the Principle of Indifference in its canonical formulation is untenable consists in the lot of paradoxes following from its acritical application.

11.9.1. A first family of paradoxes involving the Principle of Indifference concerns polyadic partitions (that is: the possibility spaces whose alternatives are more than two), and can be exemplified as follows.

A beautiful blonde enters into the railway compartment where Tom is growing weary. Ten minutes pass and he already knows she is a thirty-five-years-old and unmarried Norwegian pediatrist named “Greta”. -Is Greta single?- Tom asks himself, and for want of privileging cognitions, the Principle of Indifference teaches him that the respective probability is 50%. Analogously, if his question were -Is Greta a divorcée?- for want of privileging cognitions the Principle would teach him that the respective probability is 50%. Exactly as if his question were -Is Greta a widow?- Then, absurdly, the probability that she is a single or a divorcée or a widow ought to be 150%.

The solution is trivial. Everyone has the right to reason on the hypotheses he prefers, but if at least one of the two following conditions
- the coherence of his hypotheses
- the coherence of his reasoning
is not respected, the conclusions he draws are manifestly and completely valueless. In the case to assign \( P=50\% \) to one of the three alternatives is to assign \( P=50\% \) to the sum of the remaining two.

Since whatever assignation is based on an (objective or subjective) partition of the possibility space, and since all the theories of probability converge on the same theorems, an anti-theoremic partition is incoherent. Then, assuming that Tom has no information about the percentages of singles, divorcees and widows relative to the thirty-five-years-old and unmarried Norwegian pediatrists, either he recognizes the impossibility of an assignation, or he must appeal to some informational integration (either ruled by the Principle of Indifference or by some subjective opinion). Then, if Tom decides to follow the Principle, since the possibility space relative to “unmarried” is partitioned in “single”, “divorced” and “widow”, he must assign the same probability (1/3) to the three alternatives. And if (much more plausibly) Tom refuses the Principle and accepts the subjective integration suggested by his common sense (according to which singles are much more numerous than widows among the thirty-five-years-old and unmarried Norwegian pediatrists) he will assign to the first hypothesis a greater probability et cetera, respecting anyhow the theoremic condition \( P_1 + P_2 + P_3 = 1 \).

In other words. While

\[
P(h|k) + P(\neg h|k) > 1\]

is absurd (antitheoremic)

\[
P(h|k_1) + P(\neg h|k_1) > 1\]

is perfectly possible, provided that \( k_1 \) and \( k_2 \) are distinct statutes. And the three previous assignations can only be justified by a reference to three incompatible statutes.

11.9.2. A second family of paradoxes involving the Principle of Indifference concerns the inverse quantities and can be exemplified as follows.

In this store there are one hundred well piled heaps of different timbers: we know only that each of them weighs exactly 2 tons and that the specific weight \( sw \) of the hundred timbers varies from 0.50 to 1.00. The Principle of Indifference (in its naive formulation) assigns the same probability to any given \( sw \) of the range, therefore, taken a generic heap, its probability of having a \( sw<0.75 \) is equal to its probability of having a \( sw>0.75 \). Yet the Principle can also be applied to the volumes \( v \), which obviously vary from 4.00m\(^3\) to 2.00m\(^3\), therefore if we choose to reason on volumes, since the same probability is assigned to each of them, the probability for a generic heap of having a \( v>3.00m^3 \) is equal to the probability of having a \( v<3.00m^3 \). The contradiction is that two inferences applied to the same situation lead to incompatible conclusions, as the specific weight of a 2 ton heap whose volume is 3,00m\(^3\) is not 0.75. but 0.66.
The solution is easy. Since $sw$ and $v$ are inverse quantities, their link is not linear, but hyperbolic, so that as soon as we assume a uniform distribution of the specific weights we are assuming a non-uniform distribution of the volumes (and vice versa).

Let me insist through a concise appeal to analytic geometry:

$$v = \frac{2}{sw}$$

is the equation of a hyperbole, that is of a curve whose average abscissa, owing to strictly mathematical reasons, cannot correspond to its average ordinate.

Here too, then, the correct procedure is sufficient in order to avoid the paradox. Since every assignation must be based on (objective or subjective) cognitions supporting it, an equiprobable assignation relative to $sw$ presupposes cognitions supporting it; and these same cognitions, necessarily, sustain a non-uniform distribution of the respective volumes (or vice versa) since specific weights and volumes are non-linearly related parameters.

11.9.2.1. Indeed the hypothesis that the distribution of the specific weights is uniform between 0.50 and 1.00 has been dictated by the opportunity to deal with the paradox in its usual formulation, nevertheless it is rather implausible; here too (as in the example of the crew) the most plausible hypothesis suggests a normal distribution, that is a bell-shaped curve). Anyhow the solution holds even in this case, since as soon as we suppose the symmetry of the $sw$-curve with respect to the value $0.75 t/m^3$ (so legitimating the conclusion that the probabilities of being less and of being more are equal), we are supposing an asymmetry (a deformation) of the $v$-curve with respect to the value $3.00 m^3$ (so entailing the conclusion that the probabilities of being $<3.00 m^3$ and of being $>3.00 m^3$ are different) et cetera.

11.10. A deeper analysis is needed to evaluate satisfactorily a family of geometrical paradoxes involving the Principle of Indifference. I summarize the most renowned one, that is Bertrand’s paradox, as follows.

Given a circle of radius $R$, in condition of parametric uniformity the problem (11.xii)

$$\text{what is the probability } P \text{ that a randomly drawn chord will have a length more than } R\sqrt{3}?$$

admits three incompatible solutions. If we reason on a sheaf of parallel chords (parallel approach $S_1$) the answer is $P_1=1/2$. If we reason on the sheaf of chords rotating on a point of the circumference (polar approach $S_2$) the answer is $P_2=1/3$. If we reason on the mid-points of the chords (areal approach $S_3$) the answer is $P_3=1/4$.

Its current solution (Van Fraassen, Hajek) claims that the paradox is born by the application of the Principle of Indifference to non-linearly related parameters. In my opinion such a claim is not a solution because it does not realize I) that $S_1$ and $S_2$ are only two different instances of the same application

II) that $S_3$ is vitiated by a geometrical mistake.

11.10.1. As for I) I start from a more punctual formulation of the probabilistic problem. For the sake of simplicity the discourse is limited to a semicircle. So, with reference to Figure 11.1,

where
- $O$ is the centre of a semicircle of radius $R$
- $A$ is an external and generic point on the diametric straight of basic reference
- $X$ is the distance between $O$ and $A$ (obviously $X$ varies from $R$ to $\infty$)
- $AT$ is the tangent,
- $ABC$ is a generic straight intersecting the semi-circumference in $B$ and $C$
- $ADE$ is the straight intersecting the chord of critical length ($R\sqrt{3}$)
- $M$ is the mid-point of $DE$, therefore $AM$ is the tangent to the concentric circle with radius $R/2$
- $F$ is the intersection $OT-BC$
the probabilistic problem to solve finds in (11.xiii) what is the probability $P$ that BC is longer than $R\sqrt{3}$?

its new formulation. Thus we have specified (11.xii) by considering the sheaf of chords passing for a generic point $A$ of the plane. In such a context, $\xi$ is the parameter to which the Principle of Indifference must be applied, so establishing a polar uniformity (equiprobability) for the direction of the various chords. Therefore

$(11.xiv)$ \[ \beta / \alpha \]

is the probability that BC is longer than DE. Since elementary trigonometry teaches us that

$\alpha = \arcsin(R/X)$
$\beta = \arcsin(R/2X)$

and since the lengths of OT and OM are respectively R and R/2,

$\alpha = \arcsin(2\sin\beta)$

$(AO \sin \alpha = R = 2AO \sin \beta$; the conclusion is that, on the ground of (11.xiv),

$(11.xv)$ \[ \beta / \arcsin(2\sin\beta) \]

or indifferently

$(11.xvi)$ \[ \arcsin(R/2X) / \arcsin(R/X) \]

is the ratio expressing $P$. I wrote “indifferently” to mean that (11.xv) and (11.xvi) can be immediately and reciprocally transformed, for their discrepancy depends only on the variable we choose to express the generic position of $A$: exactly as we get (11.xv) if we choose “$\beta”$, we get (11.xvi) if we choose “$X$."

Anyhow the crucial conclusion is that $P$ depends on the position of $A$. For instance

$(11.xvii)$ when $X=R$, $\alpha=90^\circ$, $\sin \alpha=1$, $\sin \beta=1/2$, $\beta=30^\circ$, $\beta/\alpha=1/3$ (that is $P_3$)
when $X=2R/\sqrt{3}$, $\alpha=60^\circ$, $\sin \alpha=\sqrt{3}/2$, $\sin \beta=\sqrt{3}/4$, $\beta=25^\circ40'$, $\beta/\alpha=0.428$
when $X=2\alpha=45^\circ$, $\sin \alpha=\sqrt{2}/2$, $\sin \beta=\sqrt{2}/4$, $\beta=21^\circ$, $\beta=0.466$
when $X=2R$, $\alpha=30^\circ$, $\sin \alpha=1/2$, $\sin \beta=1/4$, $\beta=14^\circ30'$, $\beta/\alpha=0.483$

- when $X \rightarrow \infty$, $\alpha \rightarrow 0$, $\beta \rightarrow 0$, $\sin \beta \rightarrow 2 \sin \beta$, $\beta/\alpha \rightarrow 1/2$ (that is $P_4$).

All these instances are deduced by the same polar application of the Principle of Indifference ($\delta \xi$ constant). Thus it is mathematically proved that $S_1$ and $S_2$ are nothing but two particular cases of the same approach: while $S_2$ is nothing but the application of a $\xi$-uniformity when $A$ is a point of the circumference, $S_1$ is nothing but the application of the same $\xi$-uniformity when $A$ is a point at $\infty$, for in this case the $\xi$-uniformity implies the linear $Y$-uniformity. Therefore

- the difference between $P_1$ and $P_2$ (contrary to Van Fraassen’s and Hajek’s claim) cannot depend on the application of the Principle of Indifference to non-linearly related parameters, because both of them are simply two particular cases of the situation generically illustrated in Figure 11.1;
- the difference between $P_1$ and $P_2$ depends on the fact that while $P_1$ is the value of the function $P_\xi$ when $X=\infty$, $P_2$ is the value of the same function when $X=R$;
- there is nothing paradoxical in ascertaining that different values of a function correspond to different values of its argument.

11.10.1.1. Incidentally. The crucial conclusion that $P$ depends on a free variable is also valid if we renounce the assumption of $\xi$-uniformity for an assumption of $\xi$-non-uniformity (that is if we privilege some directions). Such a new assumption entails simply the substitution of (11.xv) or (11.xvi) by more complicated trigonometric expressions where anyhow the free variable continues occurring because $P$ continues depending on the ratio $\beta/\alpha$, and this ratio continues depending on the position of $A$.

11.10.2. Coming back to the assumption of $\xi$-uniformity I prove that $S_3$ is vitiated by a geometric mistake (claim II) of §9).

First of all I reason in compliance with $S_3$ (X=\infty). In this case OT is perpendicular to OA, the points M, F, T are aligned and M is also the mid-point of OT. If F belongs to OM, the length of the respective chord is $>R\sqrt{3}$, and if F belongs to MT, the length of the respective chord is $<R\sqrt{3}$. Therefore, since OM-MT and since the geometric situation holds for whatever basic direction (it does not depend on the specific direction OA), $P_3=1/2$.

Now I reason in compliance with $S_5$. Since each chord whose length is $>R\sqrt{3}$ ($<R\sqrt{3}$) has a mid-point internal (external) to the concentric circle with radius R/2, the probability that a chord is longer than $R\sqrt{3}$ is the probability of its mid-point being in the internal circle. And since the ratio between the areas of two circles whose radii are respectively R/2 and R is 1/4, we must conclude that the respective probability is not 1/2, but 1/4.

Here too Van Fraassen’s and Hajek’s claim is not a solution because here too both approaches follow from the same assumption of $\xi$-uniformity (the same sheaf leading to $P_3$ if we reason on intersections with OT leads to $P_3$ if we
Nevertheless, the following puzzle is pending.

For lack of a datum. In fact

(11.xviii) Let O be the centre and R the radius of a circle; in conditions of parametric uniformity, what is the probability $P$ that a chord passing for a point A at a distance X from O be longer than $R\sqrt{3}$?

is a surely determinate problem and

(11.xix) $P = \frac{\arcsin\left(R/2X\right)}{\arcsin\left(R/X\right)}$

is its unobjectionable solution (since the parameter to which the Principle of Indifference can be applied is only one, that is $\xi$, no ambiguity affects the application of the same Principle). On the grounds of (11.xix) we can compute whatever specific value of $P$ (for instance the values listed in (11.xvii)); so in particular we legitimate the perfect compatibility of $S_1$ and $S_2$, in spite of $P_1 \neq P_2$. (the indeterminateness of $S_1$ and $S_2$ is only apparent, since both of them, although tacitly, fix a distance, that is $\infty$ and R, so making computable the respective probabilistic values). Nevertheless, the following puzzle is pending.

11.11. All these considerations, strictly, could lead to the conclusion that the problem proposed by (11.xiii) is indeterminate for lack of a datum. In fact

(11.xvii) $P = \frac{\arcsin\left(R/2X\right)}{\arcsin\left(R/X\right)}$

is a surely determinate problem and

(11.xviii) $P = \frac{\arcsin\left(R/2X\right)}{\arcsin\left(R/X\right)}$

is its unobjectionable solution (since the parameter to which the Principle of Indifference can be applied is only one, that is $\xi$, no ambiguity affects the application of the same Principle). On the grounds of (11.xix) we can compute whatever specific value of $P$ (for instance the values listed in (11.xvii)); so in particular we legitimate the perfect compatibility of $S_1$ and $S_2$, in spite of $P_1 \neq P_2$. (the indeterminateness of $S_1$ and $S_2$ is only apparent, since both of them, although tacitly, fix a distance, that is $\infty$ and R, so making computable the respective probabilistic values). Nevertheless, the following puzzle is pending.

11.11.1. Let us draw the most casual and rich interlacement of straight lines over a circle, and let us compute the per cent of chords whose length is $>R\sqrt{3}$. The author who performed systematically this experiment [Jaines 1973] reports he achieved results with an embarrassingly low value of $\chi^2$, that is he achieved highly converging arithmetical values of such a per cent, that is 50% (and, modestly, I can confirm both the convergence and the value). The puzzle is manifest: we empirically give a univocal and sound answer to an indeterminate problem.

In order to overcome such a puzzle, firstly we have to realize that under a more liberal interpretation, (11.xiii) can be read as a shortened formulation appealing (implicitly, of course) to some average value of $P$. Under this more liberal interpretation, the problem proposed by (11.xiii) is no longer indeterminate, since an average of various values is a single value. Yet in order to compute the average we need a density function for the various specific positions of A, that is for the various specific values resulting from the application of (11.xix). Here too the Principle of Indifference suggests us to adopt a criterion of uniformity; yet here the suggestion is insufficient. In §10.1, once fixed a certain position of A, we applied the Principle to the various possible directions of the chords passing through A, therefore to the parameter $\xi$. But any given position of A corresponds to a precise value both of the linear parameter X and of the polar parameter $\beta$ (or indifferently $\alpha$). As such we must decide if the parametric uniformity suggested by the Principle of Indifference is a linear uniformity (the various distances X are equiprobable) or a polar uniformity (the various angles $\beta$ are equiprobable). The decision is momentous because, generally speaking, a linear uniformity does not imply a polar uniformity, and in this case it implies a polar non-uniformity. I make my point clearer through a geometrically similar example.

Let ABC be a right-angled triangle having the cathetus AB of unitary length and the angle $\gamma$ in C of 60° (therefore the hypotenuse BC is $2\sqrt{3}$ and AC is $1/\sqrt{3}$); let D be the point where the bisector of $\gamma$ intersects AB. The probability that a generic point Y of AB belongs to AD is by hypothesis 1/2 if we reason on angles, that is if we apply the parametric uniformity to the generic angle $\theta$ between CA and CY, but it is 1/3 if we reason on segments, that is if we apply the parametric uniformity to the distance AY (AD=1/3, BD=2/3). No paradox, of course. Postulating the $\theta$-equiprobability (i.e.: $\delta\theta$=constant) means postulating the equiprobability of each non-constant linear interval $\tan\theta$ on AB. Thus an example of discrepancy between polar and linear uniformity is exhibited.

Of course there are geometrical contexts where linear and polar uniformities correspond (an elementary example is a point running on a circumference and the direction of the respective radius). Yet, manifestly, Figure 11.1 shows that in the situation under scrutiny $\tan\beta$ is a non-constant linear interval. Therefore the average we obtain by applying the linear uniformity must differ from the average we obtain by applying the polar uniformity.

Concisely: wherever a linear equiprobability implies a polar heteroprobability (and vice versa, obviously), a very paradox would arise only if those two different criteria should lead us to the same probabilistic assignment.

The three following arguments tell us that in this context the linear uniformity is the right choice.

11.11.1.1. A polar uniformity entails a sort of spacial rarefaction, so to say, because as X increases a progressively longer $\delta X$ corresponds to the same $\delta \beta$ (until an infinite $\delta X$ where $\beta+\delta\beta=0^\circ$). But when we draw the mentioned interlacement of straight lines, quite independently of its position on the plane every point has the same chance of becoming a point on a randomly drawn straight line (the space we move into does not rarely).

11.11.1.2. While the empirical value is 50%, the mathematical average corresponding to a polar uniformity is nearly 46.6% (very roughly, in order to avoid any appeal to integral calculi, we can simply remark that the middle $\beta$ is 15°). The discrepancy between 50% and 46.6% is just a consequence of the above denounced linear rarefaction entailed by the polar uniformity.
11.11.1.3. The mathematical average corresponding to a linear uniformity is just 50% since it identifies with the value for $X \to \infty$ (very roughly: the mid-point of a semi-straight is at $\infty$).

11.11.2. A subtle objection runs as follows. Jaines (and Gandolfi) were mistaken, as the same (11.xvii) shows that in an actually random set of chords $P=50\%$ is an unattainable average. In fact, once the interlacement of straight lines is drawn, let us classify the set of chords by couples of non-parallel straight lines; then if $\Lambda_j$ is the intersection of the generic $j$-couple and $X_j$ is its (finite) distance from $O$, a look at (11.xvii) shows that anyhow $P_j<50\%$, therefore it is mathematically impossible to obtain 50\% as an average of values <50\%.

Reply. The objection does not hold because
- either we do not account for the couples whose intersections are inside the circumference ($x<R$), therefore actually $P<50\%$, but it concerns only a subset of the chords we have drawn,
- or we do account for all the couples, and then we must recognize that the couples of chords whose $A_j$ are inside the circumference have a $P_j>50\%$ ($P=66\%$ where $R>A_jO>R/2$, and $P=90\%$ where $R/2\geq A_jO$); therefore an average $P=50\%$ is quite possible.

11.11.3. In conclusion. Since every chord has a direction, since no chord has more than one direction, since these obvious evidences do not depend on some peculiar direction, to classify the set of chords by parallels means to exhaust it without omissions and duplications. Therefore $S_1$ is the unexceptionable approach and $P_1$ is the unexceptionable probabilistic value.

A last objection. If also from the experimental viewpoint $P_1$ is the unexceptionable probabilistic value, then $P_2$ is experimentally unjustifiable.

Reply. The geometric context leading to $P_1$ (§11.1: the most casual and rich interlacement of straight lines) is evidently unfit for $S_2$, where all the straight lines ought to intersect at a single point (belonging to the circumference, furthermore). And as soon as we draw a sheaf complying with $S_2$ we get the empiric confirmation of $P_2$. In other words: $P_1$ and $P_2$ are connected to two different ‘metrics’ (that is two incompatible geometric contexts), and it is not at all puzzling that two structurally different trials lead to different results.
12.1. I claim
(a) that what is nowadays considered the canonical Bayesian position on Hempel’s paradox is incomplete, since it examines only one among many possible alternatives;
(b) that the analysis of the examined alternative and the consequent solution are misleading
(c) that there is an exhaustive solution whose conclusions comply punctually with our intuition.

Since, in order to report the canonical Bayesian position, I make reference to Howson and Urbach’s work listed in Bibliography (from now on: H&U), I accept their notational conventions, although I dissent from a symbology where the probability is sometimes monadic (unconditional) and sometimes dyadic (conditional).

12.2. First of all, I define by
\[ W_{e,h} = P(h|e) - P(h) \]
that is, in my symbology,
\[ W_{k',k°,h} = P(h|k°&k') - P(h|k°) \]
the confirmation value of evidence \( e \) on a hypothesis \( h \), (the confirmation value of an acquirement \( k' \) on a hypothesis \( h \), given a basic statute \( k° \)) and I say that \( e \) validates \( h \) (invalidates \( h \); is uninfluential upon \( h \)), iff \( W_{e,h} > 0 \) (iff \( W_{e,h} < 0 \); iff \( W_{e,h} = 0 \)). So, according to (12.i), not only “to validate”, but also “to invalidate” and “to be uninfluential” express values of confirmation (ranging from -1 to +1). In §13.5.2.1 a representation of confirmation values will be sketched.

12.3. Hempel’s paradox is well known. Briefly. Nicod’s postulate, applied to the classical example, states that a hypothesis like
\[ \text{(12.ii) All } R 	ext{ are } B \] (all ravens are black)
is validated by evidence of something that is both \( R \) and \( B \). Then, since the unexceptionable Principle of Equivalence states that any evidence has the same confirmation value on logically equivalent hypotheses, and since by Modus Tollens (12.ii) is logically equivalent to
\[ \text{(12.iii) All } \sim B 	ext{ are } \sim R \]
evidence of a \( \sim B \sim R \) (of a grey pistol \( GP \), say) ought to validate (12.ii); a conclusion which indeed hurts our common sense.

12.4. The deep incompatibilities among the various theories of probability regard many aspects of the matter (the philosophical approach, the Principle of Indifference, the problem of inductive generalizations and so on); but, as far as I know, they do not regard the logical and mathematical structure. In particular no one denies the validity of Bayes’s theorem, since it is formally derivable from universally accepted axioms. In this sense speaking of the Bayesian position seems to me rather misleading; how could we legitimate an anti-Bayesian position? In this sense, then, I think it would be better to speak of a strictly axiomatic approach.

12.4.1. Since Bayes’s Theorem (H&U, p.21) states that
\[ P(h|e) = \frac{P(h)P(e|h)}{P(e|h)P(e) + P(\sim h)P(e|\sim h)} \]
we can formally derive
\[ P(h|e) > P(h) \text{ iff } P(e|h) > (P(h)P(e|h) + P(\sim h)P(e|\sim h)) \]
that is iff \( P(e|h) > (1 - P(\sim h))P(e|\sim h) \) that is iff \( P(e|h) > P(\sim h)P(e|\sim h) \)
therefore
\[ P(h|e) > P(h) \text{ iff } P(e|h) > P(\sim h)P(e|\sim h) \]
(usually \( P(e|h) > P(\sim h)P(e|\sim h) \), implies \( P(\sim h)P(e|\sim h) > P(\sim h)P(e|\sim h) \).

In our specific case (12.v) becomes
\[ P(h|RB) > P(h) \text{ iff } P(RB|h) > P(RB|\sim h) \]
and the formal derivation above proves that (12.v) and (12.vi) are theorems, that is unobjectionable achievements. Furthermore, since analogous derivations can be re-proposed where “>” is replaced by “<” or by “=”, we can conclude that evidence of a black raven validates (invalidates, is uninfluential upon) the hypothesis \( h \) that all ravens are black iff such evidence is more (less, equally) probable under \( h \) than under \( \sim h \). Yet Nicod’s postulate omits the conditioning clause (in the case under scrutiny it reduces (12.vi) to its first member) and as such it is valid only in those contexts where the same conditioning clause is satisfied. And simply because, roughly, the assumption that
observing a black raven is more probable if all ravens are black seems plausible (here is the second member of (12.vi)), Nicod’s postulate, roughly, seems correct. Nevertheless, simply because discordant contexts (that is contexts where the second member of (12.vi) is not valid) are possible (although less plausible), Nicod’s postulate cannot constitute a general rule for a systematic approach to the matter.

12.4.2. The general rule is inferred by (12.v) and runs as follows:

(12.vii) Evidence of something that is both X and Y validates (invalidates, is uninfluential upon) the hypothesis that all Xs are Y, iff the eventual presence of some X~Y decreases (increases, does not change) the probability of observing a XY.

In its turn, as soon as we realize that it is not necessary to restrict the evidence to the observation of a XY, we can furtherly generalize (12.vii) in

(12.viii) An evidence validates (invalidates, is uninfluential upon) the hypothesis that all Xs are Y iff the eventual presence of some X~Y decreases (increases, does not change) the probability of the same evidence

The fundamental point is that Nicod’s postulate, (12.vii) and (12.viii) do not have the same rank: while the former is a questionable supplementary proposal, (12.vii) and (12.viii) are unquestionable achievements inferred by universally accepted axioms.

12.4.3. The inversion between the roles of h and e showed by (12.v) is momentous: a problem of confirmation can be solved without any collation between the probabilities of h ante and post the acquirement of e, since it is sufficient to collate the probability of e under h and the probability of e under ~h. Therefore it is explicitly and formally ascertained that every problem of confirmation concerns the collation between two incompatible universes which differ with respect to h or ~h. It is understood that I speak of two universes without any ontological implication. We have to remember the general scenery of reference. The universe we are living in is the only real universe; but as soon as we ask ourselves about the problem under scrutiny, we must compare two incompatible configurations and the probabilities of observing a black raven in each of them.

12.4.4. In empirical practice, the probability of observing a certain individual in our real universe depends on hundreds of parameters, yet, in order not to complicate the discourse with contingencies of no theoretical moment, I assume
- that a number is assigned to every individual of the universal population to which the reasoning is referred
- that the numerical output of an opportunely programmed computer indicates the observed individual
- that the outputs are equiprobable, so that the probability of observing an individual belonging to a certain set is directly proportional to the cardinality of the same set.

These assumptions are of no theoretical moment since Hempel’s paradox can be exactly re-proposed even under them.

12.5. The intuitive understanding of my analysis can be helped by an easy representation where the various configurations are represented by circles partitioned in sectors: one circle for each configuration, one sector for each set of individuals. Let me emphasize that this representation is simply an adaptation of Eulerian circles (Venn’s diagrams); contrary to our usual ⊆, it is intrinsically extensional. Yet, since for the moment I am mainly interested in the clearness of the diagrams and of their collations, the areas of the various sectors have a merely qualitative course; the only areal dependences are
- an empty set is represented by a null sector
- although the area of a sector increases (decreases) with the cardinality of the represented set, no direct proportionality exists between areas and cardinalities.

So for instance the RB-sector of Figure 12.0

![Figure 12.0](image1.png)

![Figure 12.1](image2.png)
is greater by far than the quantitatively correct one (black ravens are not a quarter of all the individuals).

Figure 12.0 represents the basic configuration $\Omega^0$, that is the $h$-compatible configuration where all ravens are black (no $R\sim B$-sector). Of course there are many different $\sim h$-compatible configurations. In some of them the RB-set is smaller than in $\Omega^0$ (the sector representing such a set is smaller than in Figure 12.0), and as such evidence of a black raven validates $h$. In others the RB-set is greater than in $\Omega^0$ (…) and as such evidence of a black raven invalidates $h$. In the remainder the $RB$-sector is the same of $\Omega^0$ (…) and as such evidence of a black raven is uninfluential upon $h$. Therefore, since a problem of confirmation is represented by the comparison between two figures, the specification of the $\sim h$-compatible configuration to be collated with $\Omega^0$ is indispensable.

12.5.1. Once assumed that $\Omega^0$ is the basic configuration where

(12.ix) All ravens are black
the privileged configuration where
(12.x) Not all ravens are black
is suggested by the same formulation of (12.ix) and (12.x); in fact the opposition between the “all” of (12.ix) and the “not all” of (12.x) suggests the reference to a same domain of quantification. Therefore we are induced to think of a common set of ravens without involving other (and not even named) categories of individuals. In other words, under their most spontaneous interpretation (12.ix) and (12.x) propose a dilemma whose empirical solution consists in examining the colour of any raven belonging to a previously given set. Figure 12.1 represents just this configuration $\Omega_1$ which differs from $\Omega^0$ only because in $\Omega_1$ the ravens-sector of $\Omega^0$ is now partitioned in an $RB$ sector and in an $R\sim B$ sector.

Under such an interpretation, since in Figure 12.0 (where $h$) the $RB$-sector is greater than in Figure 12.1 (where $\sim h$), evidence of a black raven validates $h$.

12.5.2. If all ravens are black, all non-black individuals are non-ravens (Modus Tollens). Therefore also

(12.xi) All non-black individuals are non-ravens
is true under $\Omega^0$ (actually Figure 12.0 respects (12.xi)). Nevertheless as soon as the considerations proposed in §12.5.1 are applied to the opposition between (12.xi) and (12.xii) we realize that, contrary to (12.ix) and (12.x) where “all” and “not all” refer to the set of ravens, in (12.xi) and (12.xii) “all” and “not all” refer to the set of non-black individuals and that, therefore, the most spontaneous interpretation leads to the configuration $\Omega_2$ represented in Figure 12.2.

12.5.3. Incidentally. While (12.ix) and (12.x) are introduced by merely emphasizing new lines, (12.xi) and (12.xii) are introduced by hyperlinguistic new lines (… is true under $\Omega^0$…); furthermore they all are metalinguistically recycled (… the “all” …).

12.6. As for our problem (let me reason directly on the representation), the numerous diagrams $\Omega_i$ resulting from the various tetrapartitions of a circle in the four sectors representing $BR$, $B\sim R$, $R\sim B$ and $R\sim B$, can be classified
in compliance with their respective confirmation values, that is in compliance with the areal ratios of homologous sectors in $\Omega^o$ and in $\Omega$. So, for instance, an easy collation between Figure 12.0 and Figure 12.3 shows that the latter represents a configuration where

$$P(RB|h) > P(RB|\sim h)$$

$$P(R\sim B|h) < P(R\sim B|\sim h)$$

(in particular $P(R|h) = P(R|\sim h)$)

$$P(\sim RB|h) > P(\sim RB|\sim h)$$

$$P(B\sim R|h) < P(B\sim R|\sim h)$$

(in particular $P(B|h) = P(B|\sim h)$).

Analogously Figure 12.4 represents a configuration where

$$P(RB|h) < P(RB|\sim h)$$

$$P(R\sim B|h) < P(R\sim B|\sim h)$$

(therefore $P(R|h) < P(R|\sim h)$)

$$P(\sim RB|h) > P(\sim RB|\sim h)$$

$$P(B\sim R|h) = P(B\sim R|\sim h)$$

(therefore $P(B|h) < P(B|\sim h)$)

Figure 12.5 represents a configuration where

$$P(RB|h) > P(RB|\sim h)$$

$$P(R\sim B|h) < P(R\sim B|\sim h)$$

(in particular $P(R|h) > P(R|\sim h)$)

$$P(\sim RB|h) < P(\sim RB|\sim h)$$

$$P(B\sim R|h) = P(B\sim R|\sim h)$$

and so on.

12.7. In order to connect my analysis with the solution proposed by Howson and Urbach I need to transform unconditional probabilities into conditional ones and vice versa. This task is accomplished by the equivalence

$$(12.xiii) \quad (P(R|h) = P(R)) \iff (P(R|h) = P(R|\sim h))$$

whose formal proof follows:

$$P(R|h) = P(R) \quad \text{(protasis)}$$

$$P(R|h) = P(R\&h)/P(h) \quad \text{(definition of conditional probability; H&U (4) p.14)}$$

$$P(R\&h) = P(R)P(h) \quad \text{(from protasis and definition)}$$

$$P(R\&h) = P(R)\& P(h) \quad \text{(well known theorem)}$$

$$P(R\&\sim h) = P(R)P(\sim h) \quad \text{(idem)}$$

$$P(R\&h) + P(R\&\sim h) = P(R) \quad \text{(since $P(\sim h) = 1 - P(h)$)}$$

$$P(R|P(h) = P(R) - P(R\&h)$$

$$P(R|1-P(h)) = P(R\&h)$$

$$P(\sim h) = 1 - P(h)$$

$$P(h) = P(R\&h)/P(h) = P(R|h) \quad \text{(definition of conditional probability)}$$

therefore

if $P(R|h) = P(R)$ then $P(R|h) = P(R|\sim h)$.

Reciprocally

$$P(R|h) = P(R\&h) \quad \text{(protasis)}$$

$$(P(R\&\sim h) / P(\sim h) = P(R\&h) / P(h))$$

(definition of conditional probability)

$$P(R\&h)P(h) = P(R\&h)(1 - P(h))$$

$$(P(\sim h) = 1 - P(h))$$

$$P(R\&h) = P(R\&h)(1 - P(h))$$

$$(P(R) = P(R\&h) / P(h) = (P(R|h)) \quad \text{(definition of conditional probability)}$$
therefore
if \(P(R|h) = P(R|\neg h)\) then \(P(R|h) = P(R)\).

12.8. On the basis of (12.xiii), to assume, as Howson and Urbach do (last lines of p.100),

\[
(12.xiv) \quad P(R|h) = P(R)
\]

means to assume
\[
(12.xv) \quad P(R|h) = P(R|\neg h)
\]
therefore (I continue reasoning directly on the representation) it means to admit only configurations where the \(R\)-sector (as in Figure 12.1 and 12.3) is the same of Figure 12.0; in fact (12.xiv) and (12.xv) are contradicted by configurations where (as in Figure 12.2 and 12.4 and 12.5) the \(R\)-sector is different from Figure 12.0.

12.8.1. Yet, besides (12.xiv), Howson and Urbach (second line of p. 101: By parallel reasoning...) assume

\[
(12.xvi) \quad P(\neg B|h)=P(\neg B)
\]
so that
\[
(12.xvii) \quad (P(R|h)=P(R)) \& P(\neg B|h)=P(\neg B)
\]
is their total assumption. This means that the \(\neg h\)-configuration they oppose to \(\Omega^0\) is \(\Omega^3\) that is the configuration represented in Figure 12.3, where
- the \(R\neg B\)-sector is non-null, thus complying with \(\neg h\)
- the \(R\)-sector and the \(\neg B\)-sector are the same as \(\Omega^0\), thus complying with (12.xvii).

In \(\Omega^0\), both the observation of a black raven and of a non-black non-raven validate \(h\), as both the \(RB\)-sector and the \((\neg R\neg B)\)-sector are greater in Figure 12.0 than in Figure 12.3.

The canonical Bayesian solution accepts such a conclusion and claims it is not at all paradoxical because, since confirmation is a matter of degree (H&U, p.100), as soon as we give the raven sector its nearly infinitesimal area we realize that the \(h\)-validation by a \(\neg R\neg B\)-observation (diagrammatically: the difference between the areas of the two \(\neg R\neg B\)-sectors) is of so low a degree as to be intuitively imperceptible.

12.9. I agree with the claim that confirmation is a matter of degree, yet I disagree radically with the claim that the solution of Hempel’s paradox is a matter of degree. No doubt that the ratio \(1/N\) between ravens and individuals of the universe is very little, but anyhow it is not zero, since ravens do exist. Therefore \(MN\) observations of non-black non-ravens and \(M\) observations of black ravens ought to validate \(h\) at the same degree. Which is not the case. The discovery of a huge mafia armoury crammed with 10000 grey pistols, would be a hard stroke to organized crime, but it would not be a validation of \(h\), neither at a lowest degree; also because, if it were, it would also be a validation of the hypothesis that all dolphins are trifling or that all Messalina’s lovers were brown-haired and so on.

These considerations induce me to think firmly that the path leading to the very solution of Hempel’s paradox is not the standard Bayesian one.

12.10. In order to expose plainly what seems to me the right path, let me imagine an ornithological phenomenon entailing a configuration where (12.xvii) is satisfied. Some mutation in the genoma of ravens resulted in a new and very aggressive breed of black-and-white ravens; this phenomenon determined a chromatic disturbance in the neighbouring species such that for every raven proceeding from black to black-and-white, a magpie proceeded from black-and-white to black. The plausibility of this phenomenon is scarce, nevertheless its scarce plausibility, far from weakening my argument, supports it. In fact the solution I claim is based on the following manifest points:

a) the set of non-black non-ravens is extremely heterogeneous, since it includes pink pillows, green apples, white freezers, yellow geishas, grey pistols, blue hand-bombs, black-and-white magpies et cetera;
b) in order to satisfy (12.xvii) we must renounce the most plausible way out (that is \(\Omega^i\)) according to which the apparition of non-black ravens is an ornithological phenomenon confined to the set of ravens;
c) renouncing \(\Omega^i\) does not at all mean renouncing the conviction that the apparition of non-black ravens is an ornithological phenomenon: simply we think that the apparition in \(\Omega^3\) of non-black ravens, instead of being an ornithological phenomenon confined to the set of ravens as in \(\Omega^0\), is an ornithological phenomenon affecting also individuals which are not ravens;
d) under this condition, the most plausible belief is that the non-ravens affected by an ornithological phenomenon born in the set of ravens are individuals of the neighbouring species, since it would be grotesque to claim that a mutation in the genoma of ravens determines a magic blackening of some grey pistols or pink pillows et cetera;
e) if the grey pistols subset is not affected by the eventual appearance of non-black ravens, evidence of a grey pistol is uninfluential upon the hypothesis that all ravens are black (so legitimating also under \(\Omega^0\) my position about the mafia armoury).

12.10.1. Diagrammatically the extreme heterogeneity of the non-black non-ravens set is represented by a partition of the \(\neg R\neg B\)-sector in as many subsectors as the specific subsets. And d) tells us that the GP-subsector representing the grey pistols of \(\Omega^3\) is the same as that of \(\Omega^0\), therefore that the observation of a grey pistol is uninfluential upon the hypothesis that all ravens are black.
12.10.2. Symbolically. I recognize that if we accept (12.xvii) setting aside any consideration of plausibility, (12.xviii) 
\[ P(\neg R\neg B|h) > P(\neg R\neg B|\neg h) \]
is unobjectionably valid; nevertheless (12.xviii), far from entailing the grotesque (12.xix) 
\[ P(GP|h) > P(GP|\neg h) \]
strongly suggests 
\[ P(GP|h) = P(GP|\neg h) \]
that is, on the basis of (12.v) and (12.xiii), (12.xx) 
\[ P(h|GP) = P(h) \]
and (12.xx), through (12.i), implies \[ W_{GP,h} = 0. \]

12.10.3. My solution also explains why, under (12.xvii), ascertaining that the black-and-white vest over yonder is a football jersey provides us with a piece of information uninfluential upon \( h \), while ascertaining that the black-and-white bird over yonder is a magpie provides us with a piece of information validating \( h \).

12.10.4. Let me imagine a universe whose members are only ravens, magpies and pistols. The argument based on the degree of confirmation would fail, but the paradoxicality of the conclusion would remain. Another good reason for rejecting the standard Bayesian solution.

12.11. A final step is necessary to overcome any eventual suspicion of residuary paradoxicality. Since a grotesque claim is not a self-contradictory claim, somebody might refuse to renounce (12.xix), thus legitimating a configuration where undeniably evidence of a grey pistol validates the hypothesis that all ravens are black. Would then the paradox revive? Not at all. A paradox arises only when its disconcerting conclusion ensues from non-disconcerting premises. And if someone is so credulous to believe (or so arrogant to pretend) that a mutation in the chromatic gene of some raven also entails a counterbalancing chromatic modification in the metallurgy of some grey pistol, then he cannot consider as a paradoxical result that evidence of a grey pistol validates \( h \). Such a conclusion is simply the due consequence of the disconcerting premise (since under (12.xix) there are more grey pistols in \( \Omega^b \) than in \( \Omega_h \), the probability of observing a grey pistol is greater in the configuration where all ravens are black).

12.12. Until now, in order to follow the standard Bayesian solution, I have reasoned under (12.xvii). Yet the extrapolation of the analysis to configurations where 
\[ (P(R|h) \neq P(R|\neg h)) \]
or where 
\[ (P(\neg B|h) \neq P(\neg B|\neg h)) \]
does not present any difficulty. For instance Figure 12.4 represents a configuration (\( \Omega_4 \)) where 
\[ (P(R|h) < P(R|\neg h)) \& (P(\neg B|h) > P(\neg B|\neg h)) \]
(\( \Omega_4 \) can be justified by supposing that the mentioned very aggressive breed of black-and-white ravens carried out a massacre of the neighbouring non-black species, so promoting a proliferation of black ravens). Analogously Figure 12.5 represents a configuration (\( \Omega_5 \)) where 
\[ (P(R|h) > P(R|\neg h)) \& (P(\neg B|h) < P(\neg B|\neg h)) \]
(\( \Omega_5 \) can be justified by supposing that the aggressivity of black-and-white ravens provoked so violent a raven-phobic reaction of the neighbouring non-black species that both breeds of ravens are at the risk of extinction while the non-black neighbouring species are proliferating). And so on.

In this sense my approach, grounded on 
- the substitution of (12.vi) or (12.vii) to Nicod’s postulate 
- the partition of the \( \neg R\neg B \)–set in distinct subsets (categories) 
- the disconcerting unreasonableness of the hypothesis according to which a chromatic mutation in some ravens ought to determine a change in the colour of individuals belonging to absolutely far categories (as for instance grey pistols) 
- the non-paradoxicality of a disconcerting conclusion inferred by disconcerting premises, 
allows a systematic analysis whose results are in full accordance with our intuitive requirements.

For instance, let me take two lines to discuss (12.xxi). Since black ravens are very rare in \( \Omega_5 \), evidence of a black raven validates \( h \). On the contrary evidence of a non-black non-raven, owing to the proliferation of the non-black neighbouring species, validates \( \neg h \) if the observed individual belongs to a proliferating species (a magpie, say), while it is uninfluential if the observed individual belongs to an unaffected ornithological species (a golden eagle, say) or even more if the observed individual belongs to a totally heterogeneous category of individuals (a grey pistol, say).
13.1. Undoubtedly

(13.i) how can we prove the truth of our cognitions?

is the basic question of every gnosiology. The well known sceptic answer is that if we try to prove the truth of our cognitions we meet with
- either a vicious circle (diallelus)
- or a regressum ad infinitum
- or some apodictically postulated truth.
(Münchhausen trilemma)

and that therefore such a proof is unattainable.

I think that a more constructive position is possible.

13.1.1. The coherence of the reality we live in, thence the coherence of the cognitions constituting any actual statute \( k_{act} \) is an irremovable presupposition of our knowledge. Yet a cognition about a world does not entail the objective existence of the same world. The powerful faculties of the human mind allow the creation of fictitious worlds and therefore the possibility of referring alethic predicates to the respective fictitious statutes (as, say, \( k_{myt} \) for the Greek mythology). This notwithstanding, a peculiar rank must be recognized to \( k_{act} \): in fact a fundamental asymmetry distinguishes it from any fictitious statute. I mean that, for instance, just as

(13.ii) Socrates was one-headed

is \( k_{act} \)-true because unquestionable evidence tells us that really there was a one-headed man named “Socrates”,

(13.iii) Cerberus was three-headed

is \( k_{myt} \)-true because the Greek mythology tells us that the portals of Hades were guarded by a three-headed dog named “Cerberus”. Analogously, just as

(13.iv) Socrates was three-headed

is \( k_{act} \)-false,

(13.v) Cerberus was one-headed

is \( k_{myt} \)-false. The acceptance of “true” (or “false”) is the same in both cases: exactly as stating that (13.ii) is \( k_{act} \)-true (that (13.iv) is \( k_{act} \)-false) is stating that the piece of information under scrutiny is pro-collated (anti-collated) in the respective statute \( k_{act} \) stating that (13.iii) is \( k_{myt} \)-true (that (13.v) is \( k_{myt} \)-false) is stating that the piece of information under scrutiny is pro-collated (anti-collated) in the respective statute \( k_{myt} \). The mentioned fundamental asymmetry is that fictitious statutes are, so to say, son-statutes of the actual mother-statute; in fact, for instance,

(13.iii)* in the Greek mythology Cerberus was three-headed

and

in the Greek mythology Cerberus was one-headed

are respectively \( k_{act} \)-true and \( k_{act} \)-false because unquestionable evidence tells us that the Greek mythology tells us that et cetera..

13.2. In spite of the peculiar rank we assign to \( k_{act} \) the real existence of the respective world is not absolutely sure: the ontological dimension of such a world can only be postulated. Personally I agree with this postulate, nevertheless a margin of arbitrariness survives. The super-ascertained coherence of the huge flow of information we are immersed in from the date of our birth renders probabilistically absurd thinking that the same coherence follows from a merely casual knowledge of a universal chaos. Yet none of us can a priori refuse the hypothesis that such a coherence is born by the coherent but deceiving intervention of a Superior Will. How could I peremptorily exclude that, one day or another, I should awake in an ultra-galactic laboratory, thus realizing I am only a sphere of intelligent matter stimulated by a lot of electrodes? All my life was only a virtual experience, and a smiling God in a bright uniform asks me: And so? What about the objective existence of the ‘real’ world you believed to live in?

This non-absolute sureness, nevertheless, does not legitimate the scepticism in its canonical or modern version (Albert 1985, Chapter I, section 2), because (13.i) can be proposed even under such a non-absolute sureness.

Anyhow, for the sake of expressive simplicity, I will reason in compliance with the stone-age metaphysics represented by the most common realism (no ultra-galactic laboratory, no smiling God et cetera); and just in this restrained sense I will speak of an absolute truth.

13.3. In the current practice, the statute of reference is often omitted even where it is not the actual one. This elliptic habit may find a justification in the remark that, usually, the omitted reference is specified by the context. For instance we read (13.iii) as an elliptical form of (13.iii)* because the reference to \( k_{myt} \) is contextually
clear. Some ambiguities may arise where actual and fictitious statutes overlap. Let me suppose that, according to the
official historical report,

(13.vi)    \textit{Vae conspiratoribus!}

were the first Marco Antonio’s words after Caesar was stabbed to death. Therefore (13.vi) is the right answer for the
student questioned by his history teacher about those first words. But since, according to Shakespeare’s tragedy,

(13.vii)    Friends, Romans, countrymen

were the words under scrutiny, (13.vii) is the right answer for the same student questioned by his English literature
teacher. No contradiction affects these two incompatible answers to the same question because the two teachers refer
to different (and locally incompatible) statutes as \( k_{\text{act}} \) and \( k_{\text{Shakespeare}} \) are. Here too, obviously, any ambiguity
disappears as soon as the existence of two overlapping statutes is focused on: in fact

According to Shakespeare’s tragedy “Friends, Romans, countrymen” were the first Marco Antonio’s words after Caesar was stabbed to death

is an unquestionable \( k_{\text{act}} \)-truth.

13.3.1. Henceforth I put aside fictitious statutes (owing to their reducibility), therefore the index “\( \text{act} \)” can be
omitted. Nevertheless, since different knowers or even the same knower at different moments may have a
different knowledge of the actual world, wherever there is the risk of a potential ambiguity, the punctual reference
will be specified through indexes like “\( \text{act} \)” and “\( \text{fict} \)”. This subordination, far from being a theoretical limit, is a weapon
to refine our analyses wherever, say, a \( k_{\text{act}} \)-truth must be distinguished from a \( k_{\text{fict}} \)-truth or from a \( k_{\text{act}} \)-truth. Let me quote §8.8.4: contexts where one of the two different statutes is the speaker’s one … are peculiarly insidious. Anyhow this theme will be re-taken in Appendix 16.

13.4. An alethic procedure is always organized through the institutive, the propositive and the collative
stages. The institutive stage consists basically in the sensorial perception of the reality we live in. Once we banish
hallucinations (mirages, deliriums et cetera), the trustworthiness of our sensorial perceptions is assured. This
notwithstanding current statutes (even mine!) include false cognitions. But how can a non-hallucinatory sensorial
cognition be false?

13.4.1. My answer runs as follows. Although the interpretative process is normally unrealized because of its
naturalness and immediateness, our usual cognitions are not the brute sensory data, but the pieces of information
inferred by our interpretation of such data. Therefore banishing hallucinations et cetera assures the trustworthiness
of the brute perceptions, but of course it cannot assure the trustworthiness of the inferred pieces of information.
The little spot I saw squatted beneath the bush is not a hallucination, yet if subsequent and unquestionable acquirements
should show that it was a hare I would be compelled to admit that my previous identification was wrong.

13.5. Once the coherence of our informational flow (of our statute in progress) is assumed as the
fundamental gnosiologic milestone which must be always preserved, acknowledging that sometimes a new
acquirement results incompatible with our previous statute is acknowledging that, in order to restore the coherence,
some correction (that is the substitution of some piece of information in the respective assignation.
A momentous part of our gnosiologic life consists in assigning a degree of reliability to hypotheses. Given a certain \( h \), its degree of reliability, obviously, depends on the statute \( k \); In the example \( h \) is \textit{it is raining*} and the
acquirement \( k' \) obtained by opening the shutters, improving my previous statute \( k^0 \) (the rustle), increases the
reliability of \( h \), that is its probability \((P(h|k'^0&k') > P(h|k^0))\).

13.5.2. Let me start from the simplest possibility space, that is from a dilemma, in order to introduce a
simplified version \( \text{\textcopyright}^* \) of \( \text{\textcopyright} \). If we represent the probability of \( h \) and of \( \neg h \) (obviously linked by \( P(h|k) + P(\neg h|k) = 1 \)
on a unitary segment \( 0 \leq 1 \), we get two symmetric points, whose absolute distance \((P(h|k) - P(\neg h|k)) \) varies from 0
(the two opposite hypotheses are equiprobable, so both points are in the middle of the segment) to 1 (one of the
opposite hypotheses is true and the other false, so the two points are in the extremes).

The quantity

\[ i_h = (P_h - P_{\neg h}) / (1 - |P_h - P_{\neg h}|) \]
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(where “p_h” is an abbreviation for “P(h=k)” et cetera) is called “import (of the h-correction)” or also, picturesquely, “cost (of the h-correction)”. The import is a ratio, and since in its numerator the difference between the two probabilities does not occur in absolute value, the same import is a relative quantity (a negative cost is a gain). So

\[
\begin{align*}
p_h = 1 & \quad |h = 0 \\
p_h = 9/10 & \quad |h = 4 \\
p_h = 5/6 & \quad |h = 2 \\
p_h = 3/4 & \quad |h = 1 \\
p_h = 3/5 & \quad |h = 1/4 \\
p_h = 1/2 & \quad |h = 0 \\
p_h = 2/5 & \quad |h = -1/4 \\
p_h = 1/4 & \quad |h = -1 \\
\ldots & \ldots \\
p_h = 0 & \quad |h = -\infty
\end{align*}
\]

are some instances of the import function (whose asymptotic course is highly significant).

I claim that wherever we have to correct our statute, the rule to follow is: minimize the total import of the corrections (hence “criterion of minimal charge”). The rule, for instance, states that a correction concerning a surely true piece of information (p_h = 1) entails an infinite cost (is a logical contradiction) exactly as a correction concerning a surely false piece of information (p_h = 0) entails an infinite gain (is a logical necessity). Analogously, if two opposite hypotheses have the same reliability (p_h = 1/2 = p_~h) the import of the respective correction is null. All these instances comply with our intuition.

13.5.2.1. Exactly as in \(\mathcal{R}\) a confirmation value is represented by the (relative) areal difference between the sector representing h under k\(\circ\) and the sector representing h under k\(\circ\&k\) (that is under k\(\circ\&e\)), in \(\mathcal{R}\)\(\ast\) a confirmation value is represented by the (relative) distance between the point representing h under k\(\circ\) and under k\(\circ\&k\) (under k\(\circ\&e\)). Of course any e validating (invalidating) h invalidates (validates) ~h.

13.5.2.2. Before abandoning this topic I wish to remark that also an n-ary possibility space can be represented through \(\mathcal{R}\)\(\ast\). In this case we have to deal with a unitary segment and n points whose distances from the origin have a unitary sum because they represent the probability of the respective piece of information. Obviously any distribution is dictated by the statute of reference; therefore a new acquirement increasing some distances will decrease the others et cetera.

13.6. Let me re-approach the matter through some easy examples.

Example 1. I am walking along a lonely and narrow road. Over yonder a white car is daringly overtaken by a very fast red coupé, whose driver waves at me when he passes by. I am pleased with the acuteness of my sight because the quick reflections in his windshield did not forbid me from recognizing my old friend Bob. In the meanwhile the white car arrives, its driver stops and gets out: he too is my old friend Bob! I feel dismayed: are there two Bobs? A hypothesis I reject: if Bob were a pair of persons, after sixty years of frequentation, sooner or later my proverbial acumen would have realized so unusual a peculiarity. A hallucination? I am not drunk, anyhow I pinch myself and I force him to pinch me. No hallucination. A mistaken identity? I control his features scrutinize his identity card and ask him some particulars that only Bob can know; at the end I must reject also the hypothesis of a mistaken identity. Was then the red coupé a hallucination? I remember the roar of its engine, the Doppler effect, the quake of the ground at its passage, the birds flying away and so on. Another hypothesis I must reject: too punctual and well organized was the chain of poly-sensorial perceptions. Indeed I am facing an incoherence born by some mistaken inference from absolutely trustworthy sensorial data. And I think that all of us would overcome such an incoherence by the same correction: the driver of the coupé was not Bob. A conclusion dictated just by the criterion of minimal charge. In fact

\[
\begin{align*}
\text{(13.x)} & \quad \text{a red coupé passed me by} \\
\text{(13.xi)} & \quad \text{the driver of the white car is not the driver of the coupé} \\
\text{(13.xii)} & \quad \text{the driver of the white car is Bob}
\end{align*}
\]

have so high a degree of reliability that to replace one of them with its respective opposite would be too expensive a correction. On the other hand the low reliability of

\[
\begin{align*}
\text{(13.xiii)} & \quad \text{the driver of the coupé is Bob} \\
\text{(13.xiv)} & \quad \text{the driver of the coupé is not Bob}
\end{align*}
\]

a reasonable correction. In fact a tanned and moustached person resembling Bob might have deceived even the acuteness of my sight, while his casual gesture to chase a fly or to encourage so energetic an old walker might have deceived even the acuteness of my mind, inducing me to read it as the wave of an old friend. Briefly: as (13.xiv) is perfectly compatible with (13.x), (13.xi) and (13.xii), its replacement to (13.xiii) is the cheapest way out toward a
restored coherence. And just because such a replacement is not only the cheapest way out, but even the only reasonable one, all of us would accept it spontaneously.

13.6.1. Anyhow the same procedure rules also less univocal contexts.
Example 2. The driver of the white car too passes by and waves at me without stopping, he too resembles Bob. As (13.xii) is no longer a sure cognition, I find myself in perplexity facing a possibility space partitioned in the three incompatible alternatives (13.xii), (13.xiii) and (13.xv) Bob is neither the first nor the second driver each of them with its degree of reliability (the absurdity of the fourth alternative according to which Bob ought to be both drivers allows me to neglect it directly). The respective representation is then a circle partitioned in three sectors whose measures may be modified by any subsequent acquirement or consideration. For instance, as soon as I ponder that Bob is a prudent driver the reliability of (13.xiii) decreases, and it continues decreasing as soon as I ponder that hitherto Bob’s reservedness (and parsimoniousness?) kept him far from ostentatious cars. And as soon as I remember that few days ago Bob sprained his ankle, the highly increased reliability of (13.xv) makes it a practically irrereplaceable piece of information (that is, in \(\otimes\): the highly increased area of the sector representing (13.xv) makes it the only partition of the circle).

Only some absolutely sure new acquirement contradicting some absolutely sure previous cognition could throw me into a deeply bewildering puzzle. But just the fact that after trillions and trillions of acquirements all the (relatively rare) situations affected by an apparent incoherence have been solved through a severe re-examination of the whole context and the detection of a mistake affecting the acquirement or the cognition (therefore their only presumed irreplaceableness), just this fact, then, induces me to accept as an incontrovertible datum the coherence of the informational flow constituting my mental life (quite independently of the ontological dimension of its source). In this sense the falsity of a hypothesis results from its incoherence with respect to the informational flow (§13.8 below).

13.6.2. I recall §6.13. Making reference to Example 1, let me represent the possibility space concerning the four \(t^2\)-alternatives determined by applying the opposition between being Bob (\(=b\)) and not being Bob (\(\neq b\)) to the two drivers (\(d_1\) for the driver of the coupé and \(d_2\) for the driver of the white car). Once excluded the possibility that both drivers are Bob (alternative 4) the \(t^2\)-circle is partitioned in three only sectors whose areas, before the coupé passes me by, are determined, say, by some previous statistical pieces of information about the identity of the drivers running that road; so the measure of the alternative 1 (\(d_1=b\) & \(d_2\neq b\)) will be rather high et cetera.

When at \(t\) the coupé passes me by, the sight of its driver modifies my cognitive situation increasing the probability (the measure) of the alternative 2 (\(d_1=b\) & \(d_2=b\)). If I were absolutely sure that \(d_1=b\), I could represent my (new) \(t\)-cognitive situation by shading the sectors 1 and 3. but if I am not sure, an appeal to a (new) \(t\) diagram is necessary in order to represent through the re-partitioning technique the new assignation concerning the three mentioned alternative. In other words. Either we introduce a scale of shading intensities, or we must acknowledge that shadings can only represent the elimination of an alternative. In this sense shadings are an abbreviative technique only somewhere applicable, since the most powerful and universal representation is realized by a sequence of diachronic diagrams each of them ruled by the re-partitioning technique. Of course a hybridization of the two techniques is realizable in order to reduce the number of diagrams.

13.7. A paradigm where the opposition between cognitions and propositions is combined with the opposition between alinguistic (factual) and linguistic informational sources, comprehends four points. Opposing *factual* to *linguistic*, strictly, is an approximation, since the hearing of a voice or the reading of a text too are factual perceptions. This notwithstanding the opposition is right because a linguistic source, besides adducing the non conventional pieces of information concerning its material nature, adduces also further and conventional pieces of information, that is because the essential step does not regard the inference from the listening of modulated sounds or the sight of ink arabesques to their classification as meaningful expressions, but from the text resulting from such a classification to its meaning. Therefore, although the above examples privilege alinguistic sources, the analysis is valid even where the pieces of information are linguistically mediated, that is where they result from a semantic interpretation.

Example 3. Like Example 1, except that the driver who gets out from the white car is Tom, who tells me: “Did you see? Since he fell in love and bought a Ferrari, Bob has become a risky driver!” In a situation like this, not only is (13.xiii) no longer an incompatible assumption, but its reliability is increased by Tom’s comment, that is by a completely separate phonic utterance.

Example 4. Like Example 1, except that my wife is walking with me (better: I am walking with my wife). When the red coupé passes us by, I do not recognize its driver but I hear her exclaiming “It is Bob!” The piece of information (13.xiii) is now inferred exclusively from the semantic interpretation of sounds modulated by my wife’s voice; the reliability of (13.xiii) depends on the confidence I have in her sight (and her sincerity). Of course if I believe her, when the white car stops and Bob himself gets out, the criterion of minimal charge compels me to conclude that sometimes we confide excessively in our wives.
Example 5. My wife’s exclamation (Example 4) is verified by Tom’s comment (Example 3). I only deal with uttered sounds, but in the end I believe (13.xiii) et cetera.

13.8. Let me insist (§13.4.1). The eventual occurrence of linguistic sources (then of semantic components in the interpretative stage) is a primary factor for emphasizing the distinction between the institutive and the propositional stage; nevertheless it is a secondary factor with respect to the alethic procedure because this procedure deals with a collation between pieces of information, quite independently of the way in which such pieces of information are obtained (a conclusion confirmed also by the applicability of alethic predicates to fictitious statutes). The fact that the core of an alethic procedure is not influenced by the informational source strongly evidences the short-sightedness of a merely linguistic approach to logic. Although the usual objects of alethics are the pieces of information drawn by interpreting linguistic texts, the linguistic component is not essential for alethics.

13.8.1. Where the informational source is non-linguistic, a clean distinction between brute sensorial data and inferred pieces of information is often hampered by the naturalness and immediateness of the same inferential stage. For instance what I wrote in §13.6 (“over yonder a white car is daringly overtaken ...”) does not describe my sensorial experience; what I wrote describes my interpretation of a red and a white spots moving on a grey line. I firmly believe in that interpretation because all of my subsequent perceptions confirmed it: in fact the red spot over yonder became larger in conformity with its presumed speed and with the laws of perspective, the noise I heard corresponded to the roar of a speedily approaching powerful engine, and so on. The initial interpretation is progressively legitimated by the immense quantity of subsequent sensorial data which can be coherently inserted into the same interpretation, that is by the lack of a credible alternative interpretation. What should I have thought? that the red spot was a coupé-shaped horse whose whinnies sounded like a powerful engine and so on?

The vineyard in the fog (§5.6.1) is a perhaps more punctual example showing what I mean by “immediateness of an interpretative stage”. The sight of a near and moving shadow is the sensorial datum, the awareness of my perfect clearness of mind (everything is relative) legitimates my conviction that such a datum is not a hallucination, its interpretation as a bad giant is the inferred piece of information. Hence the sensation of impending danger and my reaction. Yet subsequent and absolutely reliable acquirements incompatible with a gigantic, anthropomorphic and hostile figure is threatening us make it an untenable hypothesis; therefore its substitution with

(13.xvi) no gigantic, anthropomorphic and hostile figure is threatening us

becomes a necessary correction. Analogously subsequent and absolutely reliable acquirements make

(13.xvii) a bee-master overalls hung on a stake are tossed by a gust of wind

a further and absolutely reliable hypothesis whose perfect compatibility with (13.xvi) restores the perfect coherence of my whole experience (indeed (13.xvi) does not imply (13.xvii), the former is achieved before recognizing that the tossed cloth is a bee-master overalls). As for the ridiculousness of my instinctive reaction, it follows from the roughness of mistaking overalls and bad giants, two rather unlike categories of individuals, particularly with respect to their faculty of threatening.

13.8.2. A specification about *alinguistic*. The alinguisticity I spoke of in §5.6.1 concerns the practical impossibility of a linguistic mediation for my initial inference from the sensorial datum. The alinguisticity of Example 1 and Example 2 concerns the (factual) way in which the various pieces of information are obtained without conditioning eventual linguistic factors in the inferential procedure.

13.8.3. The assumption according to which hallucinations et cetera are banished is not theoretically reductive, since not banishing them would only mean admitting a subordinate order of possible mistakes. In fact and by far, the usual mistakes we (at least: we sober) fall into concern the interpretative stage, not the perceptive one (as for instance the fading out of the red spot I interpreted as a coupé over yonder).

Example 6. Like Example 1, but I am under a heavy LSD-effect. The red coupé, instead of passing me by, takes off and fades out in a flash. Yet as soon as my clearness revives, the recollection of my previous LSD-assumptions supplies a coherent explanation enlightening the pleasant hallucinatory phenomenon too.

13.9. Strictly, since everyone can likewise utter fallacious and veracious sentences, the intrinsic reliability of the piece of information inferred from a linguistic source (that is the reliability of the proposition we get from the interpretation of the sentence before collating it with our statute) ought to be null. Yet this is not the current case because, usually, the context supplies a meta-information suggesting us an a priori assignation. For instance I did not witness Caesar’s murder, and by hypothesis neither did I see any indisputable Hollywood movie on the topic, nor do I know how he died; therefore when I read in a celebrated encyclopaedia

(13.xviii) Brutus stabbed Caesar

I cannot collate (the proposition adduced by) (13.xviii) with its homologous cognition in my statute. Nevertheless I assign a very high degree of reliability to (13.xviii) uniquely on the basis of the authoritativensness I recognize to the speaker as having. In fact its falsity would entail the correction of many nearly incorrigible ‘meta-cognitions’ about the trustworthiness of the propositions warranted by the same celebrated encyclopaedia et cetera (or should I believe...
in a world-wide conspiracy aimed at deceiving my opinion about Caesar’s death?). In contexts like this the standard procedure (reading the sentence, understanding the adduced proposition, collating it with its homologous cognition and on the basis of such a collation assigning the respective degree of reliability to the proposition under scrutiny) is inverted: the degree of reliability is derived from the authoritativeness of the speaker.

Of course these reciprocal procedures can be (and often are) hybridized, so that the reliability of a new proposition results from the combination of our previous cognitions and the authoritativeness of the speaker.

13.10. The millenary wisdom of natural languages grasps spontaneously the similarities among very heterogeneous referents of alethic predicates. For instance, there is a strong reason why *false*, besides its official acceptation concerning propositions, can be properly attributed to banknotes, to keels and so on. These attributions follow from the fact that in all cases something is false (fallacious) if it adduces a piece of information incompatible with the statute (quite independently of its eventual apparent compatibility).

Therefore the falsity of a banknote follows from the accurate reproduction of the authentic ones; the banknotes of Monopoli are not false simply because, as they do not counterfeit anything, none of us is induced to think that a $100 Monopoli banknote is an actual one. Yet the deceptive component characterizing a ‘truly false’ banknote is unessential. A false keel is a bar attached to the actual one (to the true one) in order to improve the nautical performances of the vessel, not in order to deceive the knower, its falsity follows from the fact that an incompetent person is induced to think that it is the spine of the hull, which it is not.

13.10.1. The criterion of minimal charge and the condition of confirmation (§12.2)

\[ W_{\text{gt}}(k,h) = P(h|k^*\&k') - P(h|k^*) \]

are strictly related. In fact if \( k' \) validates (invalidates, is irrelevant to) \( h \), by definition (§8.5.2) it increases (decreases, does not change) the import of the \( h \)-correction. In other words, the \( \text{gt-} \)reliability of \( h \) is increased by the admission in \( k_\text{gt} \) of \( h \)-corroborating acquirements because as soon as \( \sim h \) takes the place of \( h \), such acquirements too must be corrected, thus increasing the cost of the correction. On the other hand the \( \text{gt-} \)reliability of \( h \) cannot be increased by the \( \text{gt-} \)lack of \( h \)-invalidating acquirements as this lack may follow from the \( \text{gt-} \)ignorance of such invalidating elements. And indeed our actual statutes are affected by many mistakes we shall never be aware of.

13.10.2. Incidentally. The more powerful the elaborative faculties of a mind, the more meaningful *false* is; in fact, once the coherence of the informational source and the non-hallucinatory character of the sensorial perceptions are postulated, the falsity can only concern the inferences drawn by the knower. And the more articulated is an inference, the more it is probable that some of its steps are misleading.

13.11. On the basis of the above mentioned homogeneity between the two terms involved in a collation (both cognitions and propositions, obviously, are pieces of information) the scholastic aphorism

(13.xix)  Veritas est adequatio intellectus et rei

finds in

(13.xx)  Veritas est adequatio intellectus et intellectus

an enlightening paraphrase.

13.12. It is easy to detect steady points of contact relating my approach with each of the three main alethic theories. In fact

- the intrinsically relational nature of alethic predicates presupposes a collation, which is to say a correspondence between two informational entities;
- the same opposition between *k*-true* and *k*-false* is the opposition between *k*-coherent* and *k*-incoherent*;
- the pragmatic functionality is the compatibility with the flow of information.

These points of contact do not follow from a pre-programmed syncretism. They are only the consequences of a completely autonomous approach (the informational one) which, moreover, is free from the difficulties affecting the other three. I mean that

- as soon as, in accordance with the substitution of (13.xx) to (13.xix), we recognize that the *correlata* linked by the correspondence are not heterogeneous (as facts and thoughts), but homogeneous (as cognitions and propositions) the nature of the same correspondence is no longer problematic, since it reduces itself to an implication;
- to say that a piece of information may be coherent (true) with respect to many statutes is to say that some lack of information affects the collation; yet as soon as such a gap is progressively reduced through further acquirements, the number of compatible statutes is also reduced, until singling out the right one
- since the truth of a proposition is its compatibility with the actual flow of information, the pragmatic functionality of every true proposition is necessary, and as such it can be assumed as an empiric criterion of truth.
14.1. I call generically “conditional” the propositional connection

\[(14.1) \text{if } p \text{ (then) } q\]

where \(p\) and \(q\) are respectively the protasis (or antecedent) and the apodosis (or consequent).

Nowadays, as far as I know, conditionals are canonically theorized through a truth functional approach. Since this approach seems to me fragmentary and inadequate, in this chapter I propose a systematic alternative. The formalization of such an alternative leads to a rather complex paradigm; yet far from being a fault, such a complexity is quite favourable evidence. In fact (let me repeat the consideration proposed in §8.15.3), since all its voices can be supported by punctual and non-interchangeable examples, a simpler paradigm would only mean too rough a classification.

For the sake of peace I will do my best in order to adapt my terminology to the current one (and “truth functional” is an example). And for the sake of concision, for instance, I will write

\[(14.2) \text{if even (then) } <6\]

in order to mean that if a certain outcome is even then it is less than 6.

14.2. Let “\(D\)” be a variable on propositional connectives, “\(N\)” a variable on alethic values, “\(x\)” and “\(y\)” (with or without indexes) variables on propositions. An \(n\)-adic propositional connective

\(D(x_1,...,x_n)\)

is a truth function iff the alethic value \(N(D(x_1,...,x_n))\) is deducible from \(N(x_1) \ldots N(x_n)\). Therefore if \(D(x_1,...,x_n)\) is a truth-function, its alethic value does not change if we replace \(x_i\) with an \(y_i\) such that \(N(x_i)=N(y_i)\).

14.3. Let

- conjunction is a truth function
- negation is a truth function
- the conjunction of truth functions is a truth function
- the negation of truth functions is a truth function
- the only truth functions are those given above

be the recursive definition of truth functions.

Therefore

\[(14.3) \sim(p \& \sim q)\]

is a truth function (false iff \(p\) is true and \(q\) is false).

14.4. Reading \((14.3)\) as \((14.1)\) gives rise to well known conclusions in frontal contrast with our common sense.

Bob bet on 6, but the outcome was 3. I comment

If the outcome had been 2, you would have won

and I reply to his astonished glance that until we agree to read \((14.3)\) as \((14.1)\), every outcome different from 3 implies the truth of my comment (obviously a conjunction where a false statement occurs is anyhow false, and as such its negation is anyhow true).

His new astonished glance reveals that, in his opinion, logic is a perhaps esoteric but surely crazy doctrine totally incompatible with his most deep-seated convictions.

Orthodox logicians neglect these difficulties; they claim that the task of the truth functional approach is to present an alethic table for each connective, not to justify consequences entailed by reading the same connectives in a certain way. Yet their attitude eludes the very hearth of the problem, that is giving a trustworthy theoretic frame to the logic by which the current use of “if (then)” is ruled. Surely this logic does exist, since in current practice we are able to distinguish sensible and senseless conditionals. And the problem too exists: why, for instance, does reading

\[\sim(p \& q)\]

as

neither \(p\) nor \(q\)

not create any impasse, while reading \((14.3)\) as \((14.1)\) does?

My answer is that the logic of *if (then)* (exactly as the logic of *because* or of *though*) cannot be theorized through a truth functional approach, but, once more, through an informational approach. According to this approach the soundness of a conditional depends on some link between the pieces of information respectively adduced by its protasis and its apodosis. And the conclusions we reach agree perfectly with our common sense.
14.5. For the present (until §14.15) let me reason on the possibility space (Howson & Urbach 2006, §2.a) or universal set (Hajek 2003, §1) $\Omega$ constituted by the outcomes of a die. In particular while $\Omega^c$ is the possibility space related to a standard (cubic) die, $\Omega'$ is the possibility space related to a dodecahedral die. Therefore, since every outcome must present one and only one number, the possible outcomes are connected by partitive disjunctions (XOR), so that

(14.iv) either 1 or 2 or 3 or 4 or 5 or 6
is the basic statute $k^c$ for $\Omega^c$, just as
(14.iv') either 1 or 2 or 3 or 4 or 5 or 6 or 7 or 8 or 9 or 10 or 11 or 12
is the basic statute $k'$ for $\Omega'$.

14.6. Statutes play a fundamental role in the informational approach to conditionals. Generally speaking, to agree that we are reasoning under a certain statute is to agree that the pieces of information constituting such a statute enter into the knowledge we use in order to state our conditionals.

For instance, again with reference to the rolling of a die, if we assume that all we know about a certain outcome is
(14.v) Bob bet on 6 and lost
then (14.ii) is true with reference to $k^c$, but false with reference to $k'$ where also 8, 10 and 12 are possible even (yet $\neg<6$) outcomes. This dependence of the truth value on the statute of reference shows that, strictly, statements like
(14.vi) (14.ii) is true are elliptical.

Let me insist: (14.vi) is elliptical because, given (14.v), (14.ii) is $k^c$–true and $k'$–false. In the current practice we can unequivocally use elliptical conditionals only where the statute of reference is unequivocally established by the specific context (for instance when we simply speak of a die, our interlocutor is induced to think of a cubic one, not of a dodecahedral one). But of course in a formal approach we must make explicit reference to the statute, because its pieces of information are a necessary component of the implicative relations on whose grounds the paradigm is structured.

14.7. The informational approach leads to a paradigm where the various kinds of conditionals are classified on the grounds of implicative relations among the three protagonists of any conditional, that is its protasis $p$, its apodosis $q$ and its statute of reference $k$. Yet, in order to simplify the formulae of the following theorization, the statute of reference is omitted (that is: it is given as implicitly understood).

First of all let me underline two preliminary assumptions.

14.7.1. The assumption of coherence
(14.vii) $\neg(p \supset \neg q)$
avoids wasting time about conditionals like
if even, then 3
whose absurdity is immediately evident owing to the logical incompatibility between protasis and apodosis.

Without the agreed simplification concerning the omission of the statute, of course,
(14.viii) $\neg(k \& p \supset \neg q)$
should replace (14.vii). Yet such a simplification is confirmed for every formula till (14.xxxii).

14.7.2. The assumption of properness is grounded on the distinction opposing proper conditionals to spurious (improper) ones. A conditional is proper iff both its protasis and its apodosis adduce information about the same possibility space. In ® such an assumption imposes that the sectors representing the protasis and the apodosis concern the partition of a same circle.

So, normally,
(14.ix) if the croupier is a polygamist, then $\sim 3$
is a spurious conditional because, normally, his polygamy does not influence the outcomes of the die he rolls. Yet if we reason within a (whimsical) context where
- there is only one polygamous croupier
- the polygamous croupier is an ill-famed trickster
- the trick of the die he rolls forbids the outcome 3
then (14.ix) is a proper (and true) conditional; in fact under such a (whimsical) context to learn that the croupier is the polygamist is to acquire a piece of information about the possible outcomes of the die, that is a piece of information which increases the statute by erasing "or 3" from (14.iv) (or from (14.iv') if the rolled die were dodecahedral). Anyhow, according to the properness assumption we neglect spurious conditionals and whimsical contexts: only proper conditionals are the objects of the following theorization.

14.8. The piece of information $c$ such that
(14.x) $p \& c \supset q$
and

(14.xi) \( p \& \sim q \supset \sim c \)

is called “content (of the conditional)”. Informally speaking, \( c \) is the smallest piece of information which (under the statute of reference) must be added to the protasis in order to imply the apodosis. In other words, (14.x) and (14.xi) are rules that every kind of conditional must satisfy.

Let me emphasize that the content \( c \) depends not only on the protasis \( p \) and on the apodosis \( q \), but also on the statute \( k \). For instance, under our \( k^\circ \), the content of

(14.xii) if even, then 6

is

(14.xiii) \( \sim 2 \& \sim 4 \)

because (14.xiii) is the smallest piece of information which, added to (14.iv) and to the respective protasis

\( \sim 1 \& \sim 3 \& \sim 5 \)

implies exactly

6

that is the apodosis. But under \( k' \) the content of (14.xii) would be

(14.xiii') \( \sim 2 \& \sim 4 \& \sim 8 \& \sim 10 \& \sim 12 \)

because (14.xiii') is exactly the smallest piece of information which must be added to the conjunction of the statute (14.iv') and to the respective protasis

\( \sim 1 \& \sim 3 \& \sim 5 \& \sim 7 \& \sim 9 \& \sim 11 \)

in order to imply the apodosis. Henceforth the examples neglect \( k' \) and make (implicit) reference to \( k^\circ \).

14.8.1. The insertion in \( c \) of superfluous elements is precluded by (14.xi). For instance

(14.xiv) \( \sim 2 \& \sim 4 \) and the die is red

cannot be the content of (14.xii) because

(14.xv) \( \sim (\sim 2 \& \sim 4 \) and the die is red)

is the opposite of (14.xiv), but evidently while \( p \& \sim q \), that is

\( \text{even} \& \sim 6 \)

does imply the opposite of (14.xiii), that is

\( \sim (\sim 2 \& \sim 4) \)

it does not imply (14.xv) because it does not tell us anything about the colour of the die.

14.9. My central claim is that the alethic value of a conditional is the alethic value of its content. This claim can be informally but easily supported by ascertaining that, after all, the content of a conditional is the piece of information the conditional in question transmits us.

Many kinds of conditionals can be distinguished. Their classification is centred on the informational relations among \( p, \sim p, q \) and \( \sim q \).

The conditionals where

(14.xvi) \( p \supset q \)

are called “implications”.

The conditionals where

(14.xvii) \( \sim p \supset q \)

are called “para-enthymemes”.

No conditional can be at the same time an implication and a para-enthymeme: if it were, by \textit{Modus Tollens} we could derive from (14.xvi) and (14.xvii)

(14.xviii) \( \sim q \supset (p \& \sim p) \)

that is an incoherence. The conditionals where

(14.xix) \( \sim (p \supset q) \& \sim (\sim p \supset q) \)

are called “enthymemes”. Evidently, since (14.xix) refuses both (14.xvi) and (14.xvii), no enthymeme can be an implication or a para-enthymeme; therefore the set of proper conditionals is partitioned (no blank, no overlap) by the three mentioned kinds of conditionals.

In particular the enthymemes where

(14.xix) \( \sim (\sim p \supset q) \)

are called “weak (enthymemes)”, and their opposites, that is the enthymemes where

(14.xxi) \( \sim p \supset q \)

are called “strong (enthymemes)”. Of course every kind of conditional must possess all the respective requisites. So for instance the complete rule for weak enthymemes is

(14.xxii) \( \sim (p \supset q) \& \sim (\sim p \supset q) \& \sim (\sim p \supset q) \)
stating respectively that the conditional under scrutiny is coherent, and that it is neither an implication nor a paraenthymeme nor a strong enthymeme. Analogously

\[ (14.xxiii) -p \supset -q \land -p \supset q \land -\lnot p \supset -q \land -\lnot p \supset q \]

is the complete rule for strong enthymemes.

Now let me analyse the classification.

14.10. Implications are a rather trivial topic. A collation between (14.x) and (14.xvi) shows immediately that an implication is a conditional with a null content. For instance

(14.xxiv) if even, then \( g \supset e \)

satisfies (14.xvi), and indeed (14.iv) tells us that no further information is needed to conclude that (14.xxiv) is true.

14.10.1. To claim that conditionals are not truth functions is to claim that implications are not truth functions. In fact, since negations and conjunctions are truth functions and since (§ 14.9) the various kinds of conditionals are defined on the sole basis of negations, conjunctions and implications, if implications were truth functions, the recursive definition of §3 would entail that conditionals too are truth functions. Let me propose a little example validating this conclusion. Although it is true that Germany is in Europe (g), that Berlin is in Germany (b), that Kyoto is in Japan (j) and that Berlin is in Europe (e),

\[ b \land g \supset e \]

is true, but

\[ b \land g \supset j \]

is false (contrary to *Berlin is in Europe*, *Kyoto is in Japan* is not deducible from *Berlin is in Germany* and *Germany is in Europe*); therefore both true and false implications exist whose protasis and apodosis are true.

This conclusion is perfectly compatible with the informational approach, where an implication is valid (under a certain statute) iff the piece of information adduced by its apodosis is deducible (under the same statute) from the piece of information adduced by its protasis.

14.10.2. Conditionals like

(14.xxv) if \( p \) then even or odd (that is conditionals whose apodosis is necessarily true) could be read as limit implications.

14.10.3. Conditionals like

if even, then even instance a banal sort of implication where \( p = q \) (tautological conditional).

14.10.4. The existence of implications like

if the outcome is \( \geq 2 \) and not divisible by \( 3 \), it is either quadratic or prime whose understanding is not so immediate, is of no theoretical moment.

14.11. Para-enthymemes are a strained kind of conditionals where the same “if” becomes a potentially misleading occurrence, since it seems to subordinate the apodosis to a condition which such an apodosis is not at all subordinate to. For instance

(14.xxvi) if even, then \( \sim 2 \land \sim 4 \)

is an unquestionable para-enthymeme because

(14.xxvii) if odd, then \( \sim 2 \land \sim 4 \)

is an unquestionable implication. The formal proof that in a para-enthymeme (without any appeal to \( p \)) \( c \) is sufficient to imply \( q \) runs (concisely) as follows:

<table>
<thead>
<tr>
<th>Step</th>
<th>Formula</th>
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<tbody>
<tr>
<td>1.</td>
<td>( p \land c \land q \equiv p \land c )</td>
<td>(from (14.x))</td>
</tr>
<tr>
<td>2.</td>
<td>( p \land c \land \lnot q \equiv \bot )</td>
<td>(Chapter 7, Theor 10)</td>
</tr>
<tr>
<td>3.</td>
<td>( \lnot p \equiv q )</td>
<td>(rule for para-enthymemes)</td>
</tr>
<tr>
<td>4.</td>
<td>( \lnot p \land q = \lnot p )</td>
<td>(from (3), by definition)</td>
</tr>
<tr>
<td>5.</td>
<td>( p \land \lnot q = \bot )</td>
<td>(Chapter 7, Theor 10)</td>
</tr>
<tr>
<td>6.</td>
<td>( \lnot q \land q = \bot )</td>
<td>(Chapter 7, AX4)</td>
</tr>
<tr>
<td>7.</td>
<td>( \lnot q \land q = \bot )</td>
<td>(Chapter 7, Theor 11)</td>
</tr>
<tr>
<td>8.</td>
<td>( c \land \lnot q = \bot )</td>
<td>(substitution of identity (7) in (2))</td>
</tr>
<tr>
<td>9.</td>
<td>( c \land q = c )</td>
<td>(Chapter 7, Theor 11)</td>
</tr>
</tbody>
</table>

14.11.1. Conditionals like

if even or odd, then \( q \) (that is conditionals whose protasis is necessarily true) could be read as limit para-enthymemes, since the basic requisite is kept according to which the protasis does not give any informational contribute. Indeed also conditionals
like (14.xxv) can be read as para-enthymemes, since their apodosis holds quite independently on the protasis. In spite of this double reading (as limit implication and limit para-enthymemes), the specific context makes (14.xviii) an acceptable consequence, because $\neg q$ becomes $\bot$.

14.11.1.1. Conditionals like

\[
\text{if even or odd, then } <7
\]

(that is conditionals whose protasis and apodosis are necessarily true) sound as a rhetorical way to emphasize the necessary truth of the apodosis by the necessary truth of the protasis.

14.12. Enthymemes are the core of conditionals, exactly because both protasis and content play a non superfluous role. First of all I propose two examples where (always under our statute $k^\circ$), the same protasis (if even) and the same content ($\neg 2\&\neg 4$) lead to different enthymemes owing to their different apodoses.

14.12.1. The conditional

\[
(14.xxviii) \quad \text{if even, then } >4
\]
is a weak enthymeme. In fact, as

- being even does not imply being $>4$ (for instance 2 is even and $\neg >4$)
- being even does not imply being $\neg >4$ (for instance 6 is even and $>4$)
- being odd does not imply being $>4$ (for instance 3 is odd and $\neg >4$)
- being odd does not imply being $\neg >4$ (for instance 5 is odd and $>4$)

the complete rule (14.xxii) is satisfied by (14.xxviii).

14.12.2. The already proposed conditional (14.xii), that is

\[
(14.xii) \quad \text{if even, then } 6
\]
is a strong enthymeme. In fact, as

- being even does not imply being 6 (for instance 4 is even and $\neg 6$)
- being even does not imply being $\neg 6$ (for instance 6 is even and $\neg (~6)$)
- being odd does not imply being 6 (for instance 3 is odd and $\neg 6$)
- being odd implies being $\neg 6$ (no odd number can be 6)

the complete rule (14.xxiii) is satisfied by (14.xii).

14.12.3. The passage from a weak enthymeme to a strong one having the same content, is legitimated by a variation of the informational endowment entailing the refusal of $q$ when $\neg p$. And such a variation can be attained through two reciprocal paths, that is either through an increment of $q$ or through a decrement of $p$. For instance, in order to transform (14.xxviii) into (14.xii), we can

- either to follow the first path by adding $\neg 5$ to its apodosis $>4$ (the increased apodosis $>4\&\neg 5$ is exactly 6);
- or to follow the second path by ablating the same $\neg 5$ from its protasis which thus becomes $\neg 1\&\neg 3$ (evidently if $\neg 1\&\neg 3$, then $>4$)

is a strong enthymeme because the opposite of its protasis ($\neg (~1\&\neg 3)$, that is $1\lor 3$) implies the opposite of the apodosis ($\neg >4$)).

14.13. The diagrams of $k^\circ$ can efficaciously help the intuitive understanding of the classification. Of course $k^\circ$ results from the uniform partition of the circle in six sectors. So, while Figures 14.1 represents $p$, (odd sectors precluded) and Figure 14.2 represents $\neg p$ (even sectors precluded)

![Figure 14.1](image1.png)
![Figure 14.2](image2.png)

Figure 14.3 and Figure 14.4
represent respectively $>1$, and $\sim 2 \& \sim 4$, that is respectively the apodoses of the implication (14.xxiv) and of the paraenthymeme (14.xxvi), exactly as Figure 14.5 and Figure 14.6.

represent respectively $>4$ and 6, that is respectively the apodoses of the weak enthymeme (14.xxviii) and of the strong enthymeme (14.xii).

On the ground of what I affirmed in §14.8 (“c is the smallest piece of information which must be added to the protasis in order to imply the apodosis”) such a content is represented by shading in Figure 14.1 all the sectors necessary to make the shaded field of the apodosis under scrutiny completely covered by the shaded field of the new diagram. Therefore the collation between Figure 14.1 and Figure 14.3 shows that no sector must be shaded (null content of an implication) in order to achieve such a result: Analogously a collation between the pairs of the other three conditionals shows that their content is the same, since in all of them it is represented by shading the sectors 2 and 4 (that is: such a content is in any case $\sim 2 \& \sim 4$).

All the conclusions of the above analysis are punctually verifiable through the figures. Two examples.

14.13.1. According to the definition (14.xvii), (14.xxvi) is a paraenthymeme because (14.xxvii) is an implication. Such evidence is represented by the fact that the shading field of Figure 14.2 covers the shading field of Figure 14.4.

14.13.2. The diagram obtained by representing also the content, that is by shadings also sectors 2 and 4 of Figure 14.1 is exactly Figure 14.6; this means that the apodosis of the strong enthymeme (14.xii) adduces precisely the piece of information resulting from $p \& c$. The argument proposed in §14.12.3 results from a collation between Figure 14.5 and Figure 14.6, whose only difference concerns sector 5.

14.13.3. The representation of a spurious conditional would need two circles, one for the possibility space of the protasis and one for the possibility space of the apodosis. The whimsical context of §14.7.2 allows us to shade sector 3 as a consequence of the croupier’s polygamy; otherwise such a sector is virgin. In this sense spurious conditionals might be considered as a degenerate form of paraenthymemes (§14.11.1.1).

14.14. The definitions (14.xvii) (14.xx) and (14.xxi) show that the answer to the crucial question (14.xxix) and if $\sim p$? is the essential factor ruling not only the distinction between paraenthymemes and enthymemes, but also the distinction between strong and weak enthymemes. In fact the answer to (14.xxix) must be (14.xxx), then $q$ if the conditional is a paraenthymeme (as for instance (14.xxvi)), (14.xxxi) then may be $q$, may be $\sim q$
if the enthymeme is weak (as for instance (14.xviii)), and

(14.xxxii) \( \text{then } \neg q \)

if the conditional is a strong enthymeme (as for instance (14.xiii)).

14.15. Until now I reasoned on the extremely simple possibility space represented by the outcomes of a die. Yet in our minute practice we mainly use conditionals with reference to highly complex possibility spaces concerning the world we live in. In order to acknowledge the deep differences between conditionals like the above instanced ones and conditionals like

(14.xxxiii) if Jim wins the lottery prize, he will purchase a country seat

I speak respectively of analytic (or unambiguous) and of synthetic (or ambiguous) conditionals.

Since an explicit protasis and an explicit apodosis occur in synthetic conditionals too, their ambiguity depends only on their statute (and, consequently, on their content). This means, following the example, that the same (14.xxxiii) which under some statute may be read as a para-enthymeme under another statute may be read as a weak or as a strong enthymeme. In other words, a univocal classification of (14.xxxiii) may be hampered or even forbidden by the lack of more specific information about the context, that is by the impossibility of establishing a univocal statute (and, consequently, a univocal content).

The necessity of establishing a precise statute of reference in order to derive a precise content, then a precise alethic value, entails the necessity of replacing the simplified rule (14.vii) with (14.viii) and analogously the necessity of replacing (14.x) and (14.xi) with

(14.xxxiv) \( k \land p \land c \supset q \)

and

(14.xxxv) \( k \land p \land \neg q \supset \neg c \)

respectively.

14.15.1. There are many (and potentially contrasting) reasons that can induce Jim to purchase a country seat if he wins the lottery prize: the social ambition (only the owners of a country seat can ...), a damnable promise to his wife (I promise you that if I win ...), a mere financial choice (today the best investment is ...) et cetera. Yet, since (14.xxxiii) does not inform us about such reasons, I will speak of a firm intention to summarize generically all of them.

Once obviously assumed

- Jim wins the lottery prize

as \( p \) and

- Jim purchases a country seat

as \( q \), let me schematically assume

(14.xxxvi) - the lottery prize gives sufficient suitable funds for purchasing a country seat

- sufficient suitable funds and the firm intention to purchase entail a purchase

as \( k \), and

(14.xxxvii) - Jim firmly intends to purchase a country-seat as \( c \). Under these assumptions, since \( q \) (Jim purchases a country seat) is actually deducible from \( k \land p \land c \) in conformity with (14.xxxiv), no further information is necessary to make true (14.xxxiiii). Furthermore, since (14.xxxv) too is satisfied, (the lottery gave Jim sufficient suitable funds, nevertheless he did not make the purchase, then he had not the firm intention), the content of (14.xxxiii) is actually (14.xxxvii). Yet which kind of conditional (14.xxxiii) is? In compliance with (14.xxiv), that is and if Jim does not win?

is the basic question whose answer allows the classification of (14.xxxiii). And such an answer depends on the pieces of information we add to (14.xxxvi). I sketch three possible contexts.

Under \( k_1 \), Jim, besides being a very rich person who recently inherited a lot of gold bars (thus disposing of sufficient suitable funds) is a hard speculator who thinks that today the purchase of a country seat is the best investment and an ambitious man aspiring at the admission to an illustrious club where only important land owners are welcome. Under this \( k_1 \), (14.xxxiii) tends to be read as a para-enthymeme (Jim will purchase anyway).

Under \( k_2 \), Jim, besides being a very rich person who recently inherited a lot of gold bars, is an absolutely unforeseeable guy: what then will he decide about the purchase if he does not win? Under this \( k_2 \), (14.xxxiii) tends to be read as a weak enthymeme.

Under \( k_3 \), Jim is a poor and honest pensioner with no chance of borrowing money or of robbing a bank et cetera. Under this \( k_3 \), (14.xxxiii) tends to be read as a strong enthymeme.

The point to remark is that the considerations about the alethic value of (14.xxxiii), which depends on the alethic value of its respective content are quite distinct from the considerations about the alethic value of the protasis, which depends on the result of the lottery.

14.15.2. Of course the analysis above is far from exhausting the informational nuances offered by the huge complexity of the statute in which the conditional under scrutiny is inserted. For instance, let (14.xxxiii) be a gossip
Bob imparts to me; since both of us know that while Jim repeatedly declared his intention to purchase a yacht if should he win the lottery prize, Jim’s wife repeatedly declared her intention to purchase a country seat if should she win the lottery prize, in such a situation the content of (14.xxxiii) is something like *Jim is dominated by his wife*. On the contrary, if we know that while Jim’s wife repeatedly declared her intention to purchase a yacht if should she win the lottery prize, Jim (who cordially detests the country life) repeatedly lamented that his condition of a without property guy forbids his admission to the mentioned club, the content of (14.xxxiii) is something like *Jim is dominated by his social ambition*. A more sophisticated analysis could be carried out by supposing that only Bob knows the context, so that his statute and mine (therefore his content and mine) are different et cetera.

A last example follows from the hypothesis that (14.xxxiii) is a statement I hear casually, without knowing Jim; under such a hypothesis the absolute ambiguity of the respective statute (the disjunction of too many alternatives, each of them with its respective content) forbids any alethic assignation and any classification.

14.16. The truth functional connection (14.iii), that is, more generally (§7.11),

\[(14.xxxviii) \neg(h_1 \& \neg h_2)\]

is called “pseudo(hypothetic”. While an inference from the proper conditional

\[(14.xxxix) \text{ if } h_1, \text{ then } h_2\]

to the corresponding pseudo(hypothetic is always legitimate, the reciprocal inference may be illegitimate. In fact while the truth of (14.xxxix), besides assuring the truth of (14.xxxviii), that is the falsity of

\[(14.xxx) h_1 \& \neg h_2\]

assures also that \(h_1\) and \(h_2\) are two pieces of information concerning the same possibility space, no properness condition binds the truth of (14.xxxviii), that is the falsity of (14.xxx), therefore the properness of (14.xxxix) is not assured.

For instance, the truth of (14.xxxiii) entails that

Jim wins and he does not make the purchase

is false; but the falsity of

Jim wins and Kyoto is not in Japan

does not entail that

If Jim wins, then Kyoto is in Japan

is a proper conditional. Connecting two spurious statements by “if (then)” is a misleading procedure.

14.17. As to composite conditionals like

\[(14.xxxi) p_1 \supset (p_2 \supset q)\]

it is easy to ascertain that (14.xxxi) and

\[(14.xxxii) (p_1 \& p_2) \supset q\]

are equivalent; in fact

\[(p_1 \& p_2 \& q) = (p_1 \& p_2)\]

can be indifferently deduced by definition from (14.xxxii) or by substitution of identity \((p_2 \& q = p_2)\) from (14.xxxi).

14.18. Another traditional theme disqualifying the truth functional approach to conditionals (Quine 1959, §§ 2 and 3) concerns counterfactuals (i.e. conditionals whose protases are openly assumed as false). In my opinion an approach entailing that an outcome 3 makes true both

\[(14.xxxiii) \text{ if the outcome had been } 2, \text{ Bob would have won}\]

and

\[(14.xxxiv) \text{ if the outcome had been } 2, \text{ Bob would have lost}\]

is to refuse without appeal, particularly because the informational approach theorizes counterfactuals in a way which complies perfectly with our intuitive suggestions. Briefly, I explain myself by comparing

\[(14.xxxv) \text{ if the outcome is } 2, \text{ Bob wins}\]

with (14.xxxiii). The difference between moods and tenses of their verbs means just two different positions about the alethic value of the protasis (false in the counterfactual (14.xxxiii), unknown in the indicative conditional (14.xxxv)). Yet their content is the same (*Bob bet on 2*). Therefore, since in the informational approach the alethic value of a conditional is the alethic value of its content, if (14.xxxiii) is true (14.xxxv) too is true, and if (14.xxxiii) is false (14.xxxv) too is false. Reciprocally, since the content of (14.xxxiv) is the exact opposite (*Bob did not bet on 2*), if (14.xxxiii) is true (14.xxxiv) is false, and if (14.xxxiii) is false, (14.xxxiv) is true.

In other words. The transformation of an indicative conditional into the corresponding counterfactual (or vice versa) modifies the adduced pieces of information only as for the alethic value of the protasis (perhaps true or perhaps false in the indicative conditional, surely false in the counterfactual); yet, since such a transformation does not modify their common content, it does not modify their common alethic value.

14.19. Indeed it seems to me that all these conclusions agree perfectly with our intuition.
15.1. In §6 of *The principles of mathematics* Russell says that *variable* is one among the most difficult notions of Logic. From my informational viewpoint things are quite simpler: a free individual variable is a sign (a sign, I repeat) which refers to a precise member of the set constituting its domain without supplying sufficient information to single out the precise member it refers to. In other words a free variable is a sign affected by an institutional lack of information, and “the specific but here unspecified individual belonging to the domain of reference” or, roughly, “the individual to specify” are reasonable periphrases.

15.1.1. Of course even an indefinite article like “a” does not specify a precise member. Awaiting a detailed analysis, here I shortly note that the difference consists in the opposition *free* vs. *bond*, that is, for instance, the difference between *he* and *someone* (so to write, whatever he is someone, but only a particular someone is he).

15.1.2. Therefore, though constants and variables are signs (syntactical entities), their discriminating factor is semantic. The distinction between pertinence and regard (§2.9.2) shows that defining an attribute concerning signs on the grounds of a semantic characteristic is a perfectly legitimate procedure. The (indeed rather consequential) opposition between impotent and active men is grounded on certain sexual faculties, but it concerns men, not sexual faculties. A variable, so to say, is a referentially impotent term.

15.1.3. The current formal approaches conceal a lot of semantics behind *sign*. The semantic dimension is essential to something being a sign just as the matrimonial dimension is essential to someone being an ill-married man. Once castrated (I am speaking of the sign), that is once deprived of any meaning, it becomes a dead support of nothing more than itself, an arabesque devoid of any component able to make it a genuine sign.

15.2. I try to explain my viewpoint through a picturesque example. The individual constants of an emblematic language $L_e$ are exactly (metalinguistic new-lines)

\[(15.i)\]

(which in the privileged interpretation designates Jesus),

\[(15.ii)\]

(which in the privileged interpretation designates Hitler) and

\[(15.iii)\]

(which in the privileged interpretation designates De Gaulle). In $L_e$ the graphically enlighting choice for the respective variable would then be

\[(15.iv)\]

because, since (15.iv) is the graphic core common to (15.i), (15.ii) and (15.iii), any constant is obtainable by (15.iv) through a graphic completion of this core-emblem; and the semantic integration saturating the lack of information adduced by the variable is nothing but the informational consequence of such a completion. In other words: three different fixations of (15.iv) lead to (15.i), (15.ii) and (15.iii) respectively, so determining three different semantic integrations.

Indeed, in general, settling a correspondence between graphic increments (decrements) of a sign and increments (decrements) of the adduced pieces of information would be a very precious convention. And I proposed $L_e$ precisely because, except rare and ad hoc exceptions (for instance assuming “Ed” as a variable ranging over a domain
whose members are Edgar, Edmond, Edward and Edwin) a convention like this is practically unrealizable in usual languages,

15.2.1. Obviously, since

(15.v)

does not belong to \(L^e\), the fixation from (15.iv) to (15.v) is senseless; in order to overcome this senselessness we should agree that in \(L^e\) (15.v) designates someone (Mondrian, say).

15.2.2. In §15.1 I emphasized that variables are signs. Particularly where variables are concerned, any ambiguity between the syntactic and the semantic planes is a calamity. Yet *variable*, in its current acceptation, is far from being an unambiguous notion: for instance in the same article of Wikipedia I read “a variable is a value …” and “a variable is used to designate a value…”. And actually the contemporary orthodoxy considers as a variable both a variable quantity and a symbol (an expression) representing a variable quantity. But as “to be” and “to represent” are not synonyms, an ambiguity is consequently inevitable.

Before deepening this theme (§ 15.4), some (perhaps obvious) terminological agreement.

15.3. Variables range over domains (that is set of individuals). Sentences where at least one free variable occurs (free-variable-laden sentences) are open; otherwise (free-variable-free sentences) they are closed. The valence of an open sentence is the number of its different free variables. The proposition adduced by an open (closed) sentence is variant (invariant). The valence of a variant proposition is the valence of the open sentence it is adduced by. For instance

the river \(x\) is a border of state between \(y\) and \(z\)

is a trivalent sentence and

the Danube is a border of state between \(y\) and Bulgaria

is the monovalent sentence resulting from the fixation of “\(x\)” and “\(z\)”.

15.3.1. I spoke repeatedly of fixations. The syntactic component of a fixation is the replacement of a variable with a constant, that is, in \(L^e\), the graphic integration transforming (15.iv) either in (15.i) or in (15.ii) or in (15.iii)). The informational component of a fixation is the filling up of the previous lack. A more punctual lexicon can distinguish between closure (operation on signs) and saturation (operation on pieces of information).

“Conversion” is another term I shall introduce (§16.4) to mean a peculiar fixation. Of course “substitution” (§15.9) is a less punctual term, since it can also refer to a case where a variable is replaced by a different variable, therefore to a case where no saturation is achieved.

15.4. *Variable* and *abstraction* are strictly related. With reference to \(L^e\), just owing to the mentioned correspondence between graphic interventions on emblems and informational consequences, we can conceive (15.iv) as the result of an abstraction concerning the various constants, that is as the result of the operation through which we enucleate what the emblems (15.i), (15.ii) and (15.iii) have in common. In other words: if we neglect the peculiarities differentiating the single individuals of a certain set, we get a pattern that, covering the same path in the opposite direction, ramifies toward the various individuals of the domain. To abstract is to mutilate, to prune. In this sense, what Minsky (1989) calls “frame” can be called “variable”.

Also *archetype* and *common noun* are notions strictly related to *variable*. The Snake, trivially, is not the hyper-uranic snake, but the approximative image we get from the attributes characterizing any real snake (a limbless et cetera animal); in this sense the Snake is nothing but the referent of the variable ranging over the set of snakes. But, exactly as the Personage (the referent of (15.iv)), the Snake is only the necessarily vague objectification of a necessarily vague mental pattern (obviously the more comprehensive is a pattern, the less detailed it is).

Where the domain of a variable, instead of being constituted by a set of well distinguishable individuals (as the personages above) is constituted by a variable quantity (the flow of this river), the separation among the various individuals of the domain is a matter of convention (the various flows) and as such it results less evident; so the whole phenomenon is spontaneously reduced to a diachronic identity (the flow of this river), that is to a variable quantity. Nevertheless the two contexts are not separated by any theoretically momentous factor. In both of them we have a variable (a sign) a domain et cetera; the difference is that in some contexts the values are similar enough to be assumed as instances of a same and diachronically mutable individual. And the example of §15.2 shows that the use of a variable is perfectly legitimate even where there is no variable quantity.

In other words. A variable is not characterized by a peculiarity (the variability) of its referent, but by a peculiarity (the institutional incompleteness) of the information adduced by the same variable. The introduction of
(15.iv) far from influencing the fact that Jesus, Hitler and De Gaulle keep on being the only members of our universe, influences our faculties of speaking about them, giving us the possibility to generalize our discourse by leaving the individual attributes out of consideration.

15.5. Another notion I wish to compare with *variable* is *(inclusive) disjunction*. Neither of them singles out anything, yet while ascribing

(15.vi) was a dictator
to the (inclusive) disjunction of (15.i), (15.ii) and (15.iii) adds a true (then a closed) proposition because at least one of the three constants designates a dictator, ascribing (15.vi) to (15.iv) adds a variant proposition, and as such a proposition to which no alethic value can be assigned. Hintikka (1973, I.5 footnote 26) refuses to read the quantifiers as disjunctions and conjunctions, yet my opinion is different; according to common sense I think that the piece of information adduced by the existential (or respectively the universal) quantification of a variable is the piece of information adduced by the inclusive disjunction (or respectively the conjunction) of its substitutors and that, therefore, no variance occurs. The actual root of the difference between

(15.vii) Jesus or Hitler or De Gaulle was a dictator

and

(15.viii) was a dictator
even when it is agreed that the same variable ranges over Jesus, Hitler and De Gaulle) is that while (15.vii) adds *one of those personages* (then no blank), (15.viii) adds *the specific but here unspecified personage* (then an intrinsic blank). In neither case is any personage singled out, but a disjunction adds a piece of information which, so to say, is self-sufficient although it is less than the piece of information adduced by a precise constant (under the informational viewpoint, disjoining is decreasing).

In this sense I agree with Tarski’s claim that a variable is something like the blanks of a questionnaire; this notwithstanding I cannot agree with his claim that variables have no specific meaning, since such a claim mistakes the meaning for the referent.

15.6. Also a proper name acts as a variable where there is a referential ambiguity. For instance, since in our party there are three fellows named “George”, when I confide

(15.ix) George is in love
to Bob, he immediately asks me

George who?

(“who?” “what?” et cetera are the classical questions revealing the presence of a free variable; and actually, in this context (15.ix) is an open sentence, “George” is a free variable,

All Georges (of this party) are married

is a true (then legitimate) quantification et cetera. On the other hand if I confide (15.ix) to the wife of one of the Georges, owing to the privileged role her husband plays (ought to play) in the mind of a wife, “George” becomes (ought to become) a constant designating a precise individual (her George, so to write). Here too the context is an important informational source; the use of a proper name with different referents in accordance with different interlocutors is a common practice, and actually (15.ix) is a closed sentence adducing three different propositions depending on which wife I address.

15.6.1. These simple considerations overcome the problem of individual descriptions where the uniqueness condition is not satisfied; such descriptions act as variables et cetera.

15.6.2. Of course referential ambiguity is not referential poorness. Bob, roughly speaking, has three rich images of his fellows named “George”, but he does not know which of them is the one I am speaking of. On the contrary if I speak of Goedel with my mother-in-law, tacit reference is made to the only Goedel she knows, that is our enigmatic cat, while if I speak of Goedel with my barber, the only result is to put him in troubles (nearly as if I speak of Goedel with some logician I know).

15.7. The power of the informational approach is validated by its applicability to non-linguistic contexts, too. The theme will be carefully analyzed in next chapter: here I only propose a little example. The absolute silence of a crowded black-jack table is broken by the

(15.x) you are a cheat

the croupier utters staring at a precise gambler. While “you” is a constant for those who are looking at the croupier, since the direction of his eyes is the source of the informational integration necessary to single out the (presumed) cheat, the same pronoun is a variable (you who?) for those who are looking at the green baize and therefore do not perceive the stare. On the other hand if the croupier, instead of (15.x), says

(15.xi) here there is a cheat

he is quantifying (∃x(C(x))) and the proposition adduced by (15.xi) is true or false simply because it is invariant.
15.8. While the connections relating *variable* with *parameter* and with *indeterminate* are well known and unproblematic, the distinction between *variable* and *unknown* needs a more punctual analysis. Though the “\(x\)” occurring properly in

\[(15.\text{xi})\]
x is odd

occurs properly and analogously in

\[(15.\text{xii})\]
we spontaneously read \((15.\text{xi})\) as a propositional function (therefore its “\(x\)” as a very variable) and \((15.\text{xiii})\) as an equation (therefore its “\(x\)” as an unknown). Why?

Because of course (of course?) while we read \((15.\text{xi})\) as a formula adducing a proposition affected by an intrinsic lack of information (therefore as a formula adducing a variant proposition whose alethic assignation requires a previous saturation), we read \((15.\text{xiii})\) as a formula adducing a true proposition and therefore we read its “\(x\)” as a symbol used simply to mean that at first sight we are not in conditions to recognize what exact number it designates (what numeral it stands for).

The particular that while \((15.\text{xi})\) admits ‘infinite’ solutions, \((15.\text{xiii})\) admits only one, probably favours this discrepant readings; yet it is unquestionable that we can also read the “\(x\)” in \((15.\text{xi})\) as an unknown (\(x=1\) or \(x=3\) or et cetera) and the “\(x\)” in \((15.\text{xiii})\) as a variable, thus making \((15.\text{xiii})\) a propositional function (recycling use of a new line); in this case, obviously, only an ‘external’ fixation of the variable can transform the same \((15.\text{xiii})\) into an invariant proposition either true (for \(x=101\)) or false (otherwise).

The distinction between variables and unknowns is then reduced to the distinction between
- the specific but here unspecified referent
- and

the specific but here indirectly specified referent

and the similarity between the two periphrasis legitimates the use of similar symbols, so supporting the informational approach. In other words. The use of the same symbols both for variables and unknowns is justified by the common situation of interpretative perplexity; yet while the informational perplexity determined by an authentic free variable can only be overcome by an external supplement of information (the fixation), the informational perplexity determined by an unknown is overcome by the information adduced by the same unknown-laden sentence.

A certain misleading effect of the current terminology is that the unknown is (contextually) known. Anyhow I emphasize that, exactly as variables, unknowns too are signs, not ‘quantities’.

15.9. The substitution of variables is a theme to treat in meticulous detail; particularly because the above stigmatized ambiguity affecting *variable* favours the diffused confusion between substitutors and values. For instance it is rather usual to read that if “\(x\)” is a numerical variable, then numbers are its values, or that if we substitute to a variable one of its values et cetera. The two expressions are radically incompatible; the (proper) values of a variable are the members of the set constituting the domain of the same variable, and its (proper) substitutors are the names of its (proper) values. Then, for instance, while the (proper) values of a numerical variable are numbers, its (proper) substitutors are numerals.

This simple distinction is very often neglected even by celebrated authors. The first instance in my hands involves Russell himself (*The Principles of Mathematics, § 87*): in order to legitimate a statement like

every value of a variable is then a constant

we must ambiguously agree that constants are both signs and represented values, so implicitly rejecting the unambiguous agreement according to which constants are signs (naming well specified referents).

15.10. Generally speaking, the main elements in a substitution are four, and precisely
- the situation ante substitution (I call it “initial configuration” or “base”)
- what is replaced (“the substituendum”)
- what takes its place (“the substitutor”)
- the situation post substitution (“final configuration” or “resultant”).

15.10.1. A substitution can be performed on very different situations and in accordance with very different operative modalities. For instance, let the banknotes in my wallet be the initial configuration, and let me consider the three substitutions where a 20S banknote is replaced respectively by
- (a) another 20$ banknote
- (b) four $5 banknotes
- (c) one $10 banknote.

Evidently while under (a) and (b) the initial and the final configurations have equal amounts, under (a) and (c) they are different; in order to mean this aspect I say concisely that while (a) and (b) are conservative, (c) is not. My statement is concise because, strictly, the notion of conservativeness is relative to a certain criterion (the purchasing power, in this case); in fact if the criterion were the number of banknotes in my wallet, (a) and (c) would be conservative and (b) would not. In this sense the correct expression must speak of conservativeness as regards a certain criterion; and precisely because when we deal with banknotes the main criterion is their purchasing power, if the specification is wanting we are induced to think of the economic value. I agree that “\(\Xi\)-conservative” abbreviates
logic of information, p.99

"conservative as regards the criterion \( \Xi \), and that
\[
u' = \text{Subst}(u^\circ \rightarrow \text{20$/10$})
\]
symbolizes the (c) substitution, being \( u^\circ \) and \( u' \) the initial and the final configuration of my wallet.

15.10.1.1. To speak of substitutions where the initial configuration is not modified (where the original \$20 banknote is 'substituted' by the same banknote) would evidently be a terminological abuse; in fact "substitute" means *to put something in place of something else* and *else* is intrinsically incompatible with *same*. Just in this sense I remarked in §7.2.2 that "substitution of identity" is an oxymoron.

15.11. Logic is firstly interested in linguistic substitutions, that is in substitutions whose bases, substituends, substitutors and resultants are linguistic expressions.

In order to limit the analysis to the very problems themselves, I only deal with particular linguistic substitutions, and precisely with substitutions whose bases are syntactically and semantically proper atomic sentences (of course a sentence is syntactically proper if it respects the well formation rules and semantically proper if it is not affected by any sortal improperness). Furthermore the general formulation of an atomic sentence
\[ R_\eta(u_1, \ldots, u_n) \]
is epitomized in
\[ P(u) \]
where \( P \) (without inverted commas, I am not speaking of the ML-symbol in boldface, but of an unspecified object predicate as, for instance, "<(x+y)-z") is just an \( n \)-adic object relator et cetera.

Therefore a substitution (performed on a syntactically and semantically proper base)
- is syntactically proper iff substituends and substitutors have the same syntactical status
- is semantically proper iff substituends and substitutors are semantically homogeneous
- is proper iff it is syntactically and semantically proper.

So, for instance, let
\[ V(b) \]
(Bob is vegetarian) be the (true or false, but anyhow proper) initial sentence. Then
\[
\text{Subst}("V(b)" \rightarrow "V")
\]
is
\[ V(V) \]
that is a syntactically improper sentence. Analogously, once "Q" is assumed to symbolize "quadratic",
\[
\text{Subst}("V(b)" \rightarrow "Q")
\]
is
\[ Q(b) \]
that is a syntactically proper (both the substituend and the substitutor are adjectives and both of them belong to \( L \)), but a semantically improper sentence (the added attributes are heterogeneous, since *vegetarian* concerns living beings, while *quadratic* concerns numbers).

A quite peculiar kind of semantically improper substitutions are the projectively improper ones. For instance, since in
\[ (15.xiv) \quad \text{Subst}("V(b)" \rightarrow "b") \]
both the substituend and the substitutor have the same syntactical status (both of them are terms), the substitution is syntactically proper (indeed
\[ (15.xv) \quad V("b") \]
is a syntactically proper sentence). This notwithstanding, since manifestly a name ("Bob", in this case) cannot be vegetarian, (15.xv) is semantically improper. What makes it a projectively improper sentence (then what makes (15.xiv) a projectively improper substitution) is the peculiar link between the substituend and the substitutor, that is the name-relation. A projectively improper substitution occurs wherever the substitutor and the substituend belong to different dialinguistic orders (in our case the substitutor belongs to the metalanguage of the language the substituend belongs to). The example (15.xiv) has been purposely chosen in order to make evident its projective improperness: while it is rather improbable to confuse a man with his name, it is much less improbable to confuse a name with its name (we shall see in due course how insidious is such a kind of improperness).

15 12. A (linguistic) substitution is \( m \)-conservative (meaning-conservative) iff its base and its resultant adduce the same piece of information (have the same meaning). So, for instance,
\[
\text{Subst}("Jim is cowardly" \rightarrow "cowardly"/"brave")
\]
is a proper yet a non-\( m \)-conservative substitution.

An \( m \)-conservative substitution, of course, is aethetically conservative (\( \aleph \)-conservative); in fact, since aethetic values concern pieces of information, an \( m \)-conservative substitution cannot modify the aethetic value of the unmodified piece of information both the initial and the final configuration adduce.

This conclusion can be extrapolated as follows on the ground of the Theorems of Restriction and Expansion (§8.16). If the initial proposition is \( k \)-true and the substitutor adduces a piece of information implied by the piece of
information adduced by the substituend, then the resulting proposition too is \( k \)-true. Concisely: a restrictive substitution is truth-conservative. Reciprocally, if the initial proposition is \( k \)-false and the substitutor adduces a piece of information implying the piece of information adduced by the substituend, then the resulting proposition too is \( k \)-false. Concisely: an expansive substitution is falsity-conservative. Of course an \( m \)-conservative substitution is both restrictive and expansive, therefore it is both truth-conservative and falsity-conservative (that is, \( N \)-conservative).

15.13. In order to conform (provisionally) this text to the habitual terminology, in the subsequent sections truth and falsity will be directly ascribed to formulas and sentences. Therefore an expression like “the formula \( \text{so and so} \) is \( k \)-false because the described substitution does not yield the described resultant” which nevertheless is true because et cetera.

15.13.1. In general
\[ \text{Subst}(A \; v/n) \]
is the substitution performed on an \( L \)-formula \( A \), (here too without inverted commas, I am not speaking of the \( ML \)-symbol in boldface, but of an unspecified object formula as, for instance, “\( x/0 < 0 \)” by replacing the variable \( v \) (as, for instance “\( x \)” with the numeral \( n \) (as, for instance “\( 0 \)”)). The (sometimes very insidious) danger to avoid is mistaking a \( ML \)-variable like “\( v \)” (within inverted commas, now I am speaking of the symbol in boldface) with a \( L \)-variable like “\( x \)” (for instance with \( x \) as index, the latter over numbers). I wrote “sometimes very insidious” because peculiar conventions are possible (arithmetizations), under which both \( ML \)-variables and \( L \)-variables range over numbers, so masking a potentially pernicious projective improperness.

15.13.1.1. The relationship between \( ML \)-variables and \( L \)-variables is manifestly connected with the relation (§3.8.1 and §3.8.2) between *to abbreviate* (linking “Bob” with “Robert”) and *to denominate* (linking “Bob” with Bob). Since this passage is highly consequential, I will be pedantic.

Both the “\( x \)” occurring in (15.xxi) and the “\( n \)” occurring in

\[ \text{Subst}(x/0 < 0 \; x/0 \; 0); \]
is a numeral (so that any sentence obtained by erasing such variable and by writing a numeral in its place as for instance

\[ 0 < 0 \]
is odd

is a proper sentence (speaking of the number 3, in the case), the latter is a designative variable, and as such, once erased, must be replaced by the name of a numeral; and in fact

\[ 0 < 0 \]
is a numeral

is a proper sentence (speaking of the expression naming the number 3).

In other words, numerals
- are the substitutors of “\( x \)” because they name numbers and the domain of “\( x \)” is just constituted by numbers:
- are the values of “\( n \)” because the domain of “\( n \)” is just constituted by numerals.

15.13.2. I take advantage of the occasion for a quite collateral (yet theoretically momentous) consideration. In the formal approach to arithmetic the names of the various numbers are not “\( 0 \)”, “\( 1 \)”, “\( 2 \)” et cetera, but “\( 0 \)” “\( 0 \)” “\( 0 \)” et cetera. Since what does vary is the number of “\( f \)”s concatenated to a final “\( 0 \)”, the adequate choice of the respective variables ought to substitute “\( x \)”, “\( y \)” et cetera with something like “\( 0 \)” “\( 0 \)” et cetera (of course this consideration concerns metamathematics, since it is evident that the usual choice is more economic for minute mathematical uses). The theoretical importance of the mentioned new variables is the possibility of a direct (non-recursive) definition of
addition \((\xi_0 + \psi_0 = \xi\psi_0)\); and the fact that, nevertheless, multiplication could not be defined directly, offers wide room of reflection about the respective consequences on the incompleteness (Presburger).

15.14. The choice of the variables occurring in a formula is an absolutely crucial topic. For the sake of concision, let me refer to numerical variables. Since they all adduce the same piece of information (roughly: *the number to specify*), the choice of "\(x\)" or "\(y\)" et cetera is arbitrary under many aspects. Yet as soon as the complexity of the formula increases, that is, as soon as the numbers to specify are more than one, the choice is conditioned by the necessity to respect the identity and non-identity among the various unspecified numbers we are speaking of through the identity and non-identity among the respective variables. Therefore in order to avoid undue interferences and highly misleading conclusions our choice must comply with the two following rules:

**R1:** not to choose the same variable for non-necessarily identical numbers (or, worse, for necessarily non-identical numbers)

**R2:** not to choose different variables for necessarily identical numbers.

15.14.1. The two rules are nearly obvious, yet I support them by some example. Since both

\[(15.\text{xix}) \quad x/y=z\]

and

\[(15.\text{xx}) \quad x/w=z\]

say generically that a certain number is equal to the ratio of two others, (15.\text{xix}) and (15.\text{xx}) are interchangeable formulations. On the other hand

\[(15.\text{xxi}) \quad (x=y) \supset (x/y=z)\]

symbolizes correctly an elementary mathematical truth, and (15.\text{xix}) is the apodosis of (15.\text{xxi}); this notwithstanding the substitution of (15.\text{xx}) to (15.\text{xix}) in such an apodosis would lead to

\[(15.\text{xxii}) \quad (x=w) \supset (x/w=z)\]

and (15.\text{xxii}) is an evidently unacceptable formula because "\(w\)" does not witness the necessary identity with the number indicated by "\(y\)" in the protasis, that is because **R2** is violated. Reciprocally

\[(15.\text{xxiii}) \quad (x=y) \supset (x/y=y)\]

is an evidently unacceptable formula because the third occurrence of "\(y\)" in the apodosis of (15.\text{xxiii}) entails a false identity between the divisor and the result of the division, that is because **R1** is violated.

Let me insist through a comprehensive example. While

\[\exists u (u<y)\]

and

\[\exists u (u<x)\]

are interchangeable formulations

\[(x+y \neq x) \supset \exists u (u<y)\]

is a true conditional, but

\[(15.\text{xxiv}) \quad (x+y \neq x) \supset \exists u (u<x)\]

is false, exactly because the last occurrence of "\(x\)" violates both **R1** and **R2**; in fact, with reference to the last variable occurring in (15.\text{xxiv}), whatever choice different from "\(y\)" violates **R1** because it does not witness an actual identity with the "\(y\)" occurring in the protasis, moreover the choice of "\(x\)" violates **R2**, too, because it entails an abusive identity with the "\(x\)" already occurring in the protasis.

15.15. The well known condition ruling the substitutions where variables occur (the substitutor must be free for the substituendum) is usually justified by an empirical procedure, that is by exhibiting examples where some absurdity is derived from the violation of the mentioned condition (cf. for instance Kleene 1971; §18 Example 8; Hermes 1973, IX §1; Shoenfield 1967 §2.4). The informational approach allows a theoretical explanation, just by remembering the necessity to avoid undue identifications. In fact if the substituendum occurs free in the scope of a quantifier concerning a variable occurring in the substitutor, then the substitution would violate **R2** by establishing an undue identity (that is an identity absent in the initial formula). The formal procedure to avoid such a risk, that is the re-denomination of the bound variables (Hermes, ibidem IX §2) is simply the way to restore the respect of **R2**.

The fundamental conclusion is that a substitution may restrict the arbitrariness in the choice of variables.

15.16. The informational approach to *variable* allows also the derivation of

\[(15.\text{xxv}) \quad \exists x P(x)\]

from

\[(15.\text{xxvi}) \quad P(a)\]

and the derivation of (15.\text{xxvi}) from

\[(15.\text{xxvii}) \quad \forall x P(x)\]

because (15.\text{xxv}) is a restriction of (15.\text{xxvi}) and (15.\text{xxvi}) is a restriction of (15.\text{xxvii}), therefore the truth of (15.\text{xxvi}) implies the truth of (15.\text{xxv}) and the truth of (15.\text{xxvii}) implies the truth of (15.\text{xxvi}).

An analogous achievement is §17.5.2.
16.1. Just in the moment when the object seems to be caught in its immediateness, it dissolves in the total interchangeableness of every single in front of the abstractive power of the language, which then comes forth as the exclusive possible object of itself: it is the Spirit that reveals itself in the linguistic act just as the Word reveals himself in the Incarnation.

More or less (I quote by memory) this is Hegel’s subduing opinion on indexicality. Of course I will face the matter from a much more trivial (but a little less metaphysic) viewpoint. In order to avoid superfluous distinctions, I assume that the language under scrutiny is free from any homonymy and any metaphoric use.

16.2. Indexicality is a very important but a rather neglected matter. I call “indexical” an expression adducing a piece of information which depends on the context, and I call “absolute” a non-indexical expression, that is an expression adducing a piece of information which does not depend on the context. So, for instance, “aborigine”, “your”, “here” are indexical expressions, “Papuan” “owned by Ronald Reagan”, “spherical” are absolute expressions. “Papuan” is absolute because (roughly) it will always and anyhow mean an individual born in Polinesia, “aborigine” is indexical because it will mean a Papuan if the discourse concerns Polynesia, precisely as it will mean an Eskimo if the discourse concerns Greenland and so on (that is: “aborigine” adduces a piece of information which depends on its contingent utterance).

16.2.1. As far as I know, indexical expressions are present in every natural language. The reason is that the basic problem of any communication is to find an informational ground shared by speaker and interpreter; and undoubtedly the same context (in particular the same utterance), is often the most immediate common evidence.

16.2.2. Ellipses (omissions) too can have an indexical import. For instance

it is raining

is a doubly indexical expression; but while the when is adduced by the verbal tense (*is* entails *in the moment of the utterance*) the where is adduced by an omitted "here".

16.3. Indexical expressions are strictly related to variables and unknowns, since they too adduce a lack of information (henceforth: a blank). The reason why the indexical expressions we usually read do not communicate blanks, but plainly understandable pieces of information is that usually they are inserted in a context able to saturate (§15.3.1) the same blanks.

Two kinds of indexicality (the textual and the deictic ones) can then be immediately distinguished: in textual indexicality the contextual source saturating the blank is linguistic, in deictic indexicality it is extra-linguistic since it depends on some contingencies of the utterance (the speaker, the interlocutor, the moment, the place et cetera). For instance, in Tom’s comment,

(16.i)  Yesterday Bob dismissed all his grooms, and tomorrow they will apply to their trade-union seven indexical expressions occur (“yesterday”, “his”, “tomorrow”, “they”, “their” and the two verbal tenses). Yet I understand perfectly his message because all the seven blanks are saturated through the context. Three of them are promoted to absoluteness by a grammatical antecedent, then by a textual source of information (His of whom? Of Bob. They who? His grooms, therefore Bob’s grooms. Their of whom? Of Bob’s groom), and the remaining four are promoted to absoluteness by extra-textual (deictic) informational sources that is the knowledge of the moment Tom speaks (Yesterday when? The day before the day (16.i) is uttered et cetera).

For the sake of completeness (incidentally: in my lexicon, “sake of completeness” and “boring pedantry” denote the same attitude, the only difference is that the former refers to myself and friends, the latter to my opponents), then for the sake of completeness I note that the classification could be refined, for instance by distinguishing a chrononymic indexicality (“tomorrow” ...) from a toponymic indexicality (“here” ...) and so on.

16.3.1. Obviously an indexical expression appealing to an unattainable source of integrative information fails in its promotion to absoluteness. For instance while

(16.ii) we are sinking at 7°22’East, 42°38’North

is a sensible radio mayday,

(16.iii) we are sinking at 0908GMT, February the 14, 1978

would be an astonishing stupidity. In fact a radio broadcast does deictically provide the when, not the where, and as such the broadcasted text, as in (16.ii), must provide the unknown where; on the contrary in (16.iii) the when is both deictically and textually provided, but the where remains unknown.

16.4. I call
“conversion” a fixation performed through the context;
“effective (for an indexical expression)” a context iff it allows the conversion of such an indexical expression;
“defective (for an indexical expression)” a context iff it is not effective for such an indexical expression.

For instance, if Tom tells me

(16.iv) He is an unforeseeable guy
when we are speaking of Hildegard von Bingen along a desert road, the context is defective (he who?); on the contrary if Tom utters (iv) when we are speaking of Bob or when Bob itself is hailing us from his magnificent coupé, the context is effective because the blank is deictically saturated (he who? the protagonist of our present attention, that is Bob).

16.5. The informational approach, thanks to its distinction between textual and deictic indexicality, shows that Quine’s claim (also Dowty, Wall, Peters, 1989 p. 68) according to which in ordinary languages variables are pronouns without any grammatical antecedent is highly reductive.

Let me get it straight. Currently the anaphoric function is the function selecting the grammatical antecedent of an indexical expression. Yet, as soon as deictic indexicalities too are considered, no grammatical antecedent does exist, therefore the same notion of an antecedent must be widened, and the anaphoric function becomes the function selecting the integrative informational source through which the blank adduced by the indexical expression under scrutiny is saturated. A first example is the already mentioned:

you are a cheat

(§15.7). Since the same “you” which acts as a variable (you who?) for those who are looking at the green baize singles out a specific gambler for those who are looking at the croupier, the source of the informational integration necessary to convert the pronoun, therefore its antecedent, is exactly the direction of his stare.

A second example. Both the

Bob presented me with a nice manual drill; in spite of its antiquity, this drill is still a very efficient tool

tell me when we are walking far from any drill, and, two hours later, the

This drill would not bore even butter!

Tom himself tells the devil, when he is trying in vain to bore a thin wooden wall using the just mentioned very efficient tool, are indexical statements; in both of them the conversion of “this drill” is effective (no doubt about the drill Tom is speaking of). Yet while in the former case the source of information on whose basis the conversion has been performed is textual (Tom’s previous words describing Bob’s present) in the latter it is deictic (the damned tool Tom has in his hands); therefore the anaphoric function appeals to a non-linguistic (ostensive) antecedent.

The existence of non linguistic antecedents is a strong piece of evidence supporting the power of the informational approach. Anyway from now on I only deal with the textual indexicality, thus the formal exigency to have expressions as saturating sources is satisfied, and the use of “grammatical antecedents” to mean such sources is etymologically justified.

A pedantry. An antecedent can also be subsequent to and far from ‘its’ indexical expression. For instance the grammatical antecedent of the various “I” occurring in a long letter is its signature, then the last word.

16.5.1. Once the approach is well understood, translating syntactic formulations into semantic ones is an easy task. For instance saying that a free variable cannot be fixed through a substitutor where the same variable occurs free, is saying that an informational blank cannot be saturated through a piece of information affected by the same blank.

16.5.2. The attempt at formalizing the anaphoric function on mere syntactical criteria is utopia itself. In fact a minute example shows that the selection of the right antecedent may require semantic considerations, too. The syntactical structure of

(16.v) Yesterday Dan killed Ted, and today he has been arrested
is exactly the same of

(16.vi) Yesterday Dan killed Ted, and today he has been buried
nevertheless what we actually understand by reading (v) and (vi) is that Dan has been arrested and Ted has been buried; therefore, since the antecedent of “he” is the subject “Dan” in (v) and the complement “Ted” in (vi), it is impossible to formalize on mere syntactical data a conversion depending on “to be arrested” and “to be buried”.

I think that, in order to theorize the rules of conversion, linguists ought not to neglect the criterion of interpretative collaboration (the principle of charity), thus joining the syntactical and the semantic approach. Fortunately accomplishing this theorization is not necessary in order to prosecute my discourse, since the sentences of specific interest (the paradoxical and the para-paradoxical ones) are not affected by any doubt about the conversion of their variables. Anyhow, awaiting deeper considerations, we can agree that the right antecedent is the nearest available term, assuming that a term is available iff the resulting statement is not senseless (a corpse cannot be arrested).

16.6. Let me propose a detailed example which will be quite useful also in due course. I call

- “far” (symbolically “F”) a couple of towns x and y iff the distance d between them is more than 500 kilometres
- “far from y” (“Fy”) a town x iff the respective couple is far
- “barbarous” (“B”) a town iff it is far from Athens (“F_A”)
- “peripheral” ("P", provisionally) a town iff it is far from its capital.

Then the respective symbolic definitions are

\[(16.\text{vii}) \quad F(x,y) \leftrightarrow d_{xy} > 500\]
\[(16.\text{viii}) \quad F_β(x) \leftrightarrow F(x,y) \leftrightarrow d_{xy} > 500\]
\[(16.\text{ix}) \quad B(x) \leftrightarrow F_α(x) \leftrightarrow F(x,y) \leftrightarrow d_{xy} > 500\]

and, provisionally,

\[(16.\text{x}) \quad P(x) \leftrightarrow d_{x,β\odot} > 500\]

(where \(β\) is the function from a town to its capital).

Under a \(k\) stating the position, the nationality and the rank of any town in object, all the above attributes are perfectly meaningful and the respective proposition are perfectly valuable. For instance Toledo is far from Lyon, barbarous and non-peripheral, Cadiz is non-far from Sevilla, barbarous and peripheral, Piraeus is far from Paris, non-barbarous and non-peripheral. In the above definitions replacing barbarous and non(peripheral, Cadiz is non(far from Sevilla, barbarous and peripheral, Piraeus is far from Paris, non(peripheral.

The highly deceiving factor in (16.\text{x}) is the omission of a free variable in the definiendum (here is the reason why I wrote “provisionally”). While such an omission is manifest in (16.\text{xii}) because without the second fixation we face an invariant (a variant) piece of information.

Absolute and indexical predicates are congenitally incompatible because the formers (the later) adduce an indexicality. Such a correction can be carried out through two different ways, the pertinential and the variational ones:

- under the pertinential way we impose that *peripheral* is an absolute attribute, and then we must accept that it pertains to single towns
- under the variational way we impose that *peripheral* pertains to single towns, and then we must accept its indexicality.

What we cannot impose is that *peripheral* is an absolute attribute pertaining to single towns, because these two assumptions are logically incompatible. Symbolically,

- under the pertinential approach we must appeal to a notation like

\[(16.\text{xii}) \quad P(x, β(\xi)) \leftrightarrow d_{x,β\odot} > 500\]

where "P", representing an absolute attribute, does not need any index,

- under the variational approach we must appeal to a notation like

\[(16.\text{xiv}) \quad P_{β\odot}(x) \leftrightarrow d_{x,β\odot} > 500\]

where "P" is replaced with "P_{β\odot}") just in order to witness formally the indexicality of the defined attribute.
Of course both (16.xiii) and (16.xiv) respect the formal necessity to present in the definiendum the \( \beta(x) \) occurring in their common definiens. And actually, as soon as we put

\begin{equation}
  \quad y = \beta(x)
\end{equation}

we can immediately derive (16.xiii) from (16.vii) and (16.xiv) from (16.viii) by the substitution of identity (16.xv).

The general conclusion (meta-conclusion?) is that every attempt at operating formally on symbols ought to be preceded by an extremely careful critical analysis intended to reduce the risk of misleading notations. And the acritical omission of a free variable is probably the worst one. However the whole matter will be better probed in Chapter 17, thanks to the introduction of the reflexive variable.

16.6.1. If we, in analogy with the relation between “noun” and “pronoun”, agree to call “pro-adjectives” the indexical ones, “peripheral” (under the privileged variational approach) is a pro-adjective: exactly as “she” stands for “Ava”, “Eve”, “Jane” and so on according to the woman we are speaking of, “peripheral” stands for “far from Madrid”, “far from Paris”, “far from Rome” and so on according to the town we are speaking of.

16.6.2. The immediate recognition of indexical attributes may be difficult. For instance, while *infanticide* is absolute, *uxoricide* is indexical. In fact while in order to become an infanticide any infant is an adequate victim (the variable is quantified), in order to become an uxoricide the killer must join profit and pleasure, that is he must apply to his own wife.

16.7. The reason why (§5.9) I cannot accept the argument based on indexicality as evidence supporting the semantic pertinence of alethic attributes is then clear. Briefly. Since an indexical expression can be conceived as a variable standing for different absolute expressions in conformity with different specific contexts, different alethic values are not incompatible with a syntactical pertinence.

16.8. The predicates which the logical paradoxes are built on (“Richardian”, “non-self-applicable” et cetera) are indexical; realizing their indexicality is the first and crucial step to achieve the general solution. Yet since till now their indexicality, as far as I know, has not been recognized, speaking of a crypto-indexicality seems legitimate to me.

16.8.1. Extensionally, a set is indexical iff it is the extension of an indexical attribute; so, for instance, Russell’s set is indexical. Generally reasoning, statements about indexical attributes are immediately translatable into statements about indexical sets once the intensional approach is replaced by the extensional one.
A16.1. Here I wish to tackle (and to solve) some well known puzzles (grandiloquently: paradoxes): all of them are born by the violation of the competence condition, according to which (§8.8) an inference is valid in a statute iff every piece of information employed in such an inference belongs to such a statute. And since statutes concerning distinct persons or the same person in distinct moments may differ, we can accordingly distinguish two kinds of violations: the individual and the chronological ones.

A16.2. The puzzles of intentional identity do not involve the chronological dimension, and consequently the chronological index can be omitted in the respective symbolization (so we can speak elliptically of a $k_g$ instead of a $k_{g,t}$).

I tackle them making reference to Edelberg 1986; by “Ed(1)”, “Ed(2)” et cetera I recall respectively the formulae (1), (2) et cetera of the mentioned paper. In order to facilitate the collations between his formulae and mine, I accept his notations even where (for instance his use of “$E$” for “$E'$”) more meticulous conventions might be preferable. So, for instance $B_g[Ex(Px)]$ means that $g$ (at any moment $t$ interested by the reasoning under scrutiny) believes there is an $x$ being $P$, or, in other words, that $Ex(Px)$ is a datum (a piece of information) belonging to $k_g$.

A16.2.1. I epitomize the first puzzle as follows. While (Detective) A believes Smith and Jones have been murdered by two different persons, (Detective) B believes they have been murdered by the same person. Since (p.13) A and B discussed the case at length, and since both of them ignore the specific person who murdered Smith, we can assume murdering Smith as a common identifying connotation.

In Edelberg’s opinion (Ed(19) and Ed(20), p.13) the puzzle is that, though (A16.i) A believes someone murdered Smith and B believes he murdered Jones and (A16.ii) B believes someone murdered Jones and A believes he murdered Smith are symmetric (i.e., though (A16.ii) is the commutation of (A16.i) and commutability is an universally accepted property of conjunction), (A16.i) is true and (A16.ii) is false. Such a commutation is even more evident if, once any existential import is dropped from quantifiers, we formalize (A16.i) in (A16.iii) $Ex[B_g(Sx) \& B_a(Jx)]$ (Ed(21), p.14) and (A16.ii) in (A16.iv) $Ex[B_a(Jx) \& B_g(Sx)]$ (Ed(22), ibidem). The use of the same variable both within the scope of $B_a$ and of $B_g$ is justified by the mentioned common identifying connotation; $x$ is simply the otherwise unknown person who, both in A’s and B’s opinion, murdered Smith.

A16.3. My claim is that the puzzle is only a superficial fallacy caused by a mistaken choice of the antecedent converting the indexical variable “he”. For the moment, let me reason in the ordinary language.

A16.4. Claiming that (A16.i) and (A16.ii) are respectively true and false presupposes that in both of them the anaphoric function succeeds in individuating a well specified referent for “he”. So, with reference to (A16.i), (A16.v) who is he? or in other words (A16.vi) who is the person B believes murdered Jones? is our basic question. Once left out of consideration that B suspects A, the possible answers to (A16.v) are three, and precisely (A16.vii) he who actually murdered Smith (A16.viii) he whom A believes murdered Smith (A16.ix) he whom B believes murdered Smith.

In order to prove that (A16.vii) is the only correct answer, let me consider an interim similar context where - Smith has been actually murdered by his secretary Charlie - A believes that Smith has been murdered by his barber Donald - B believes that Smith has been murdered by his pedicure Edward.

Under this context
A believes Donald murdered Smith and B believes he murdered Jones is unquestionably false. In fact ((A16.x) tells us that B believes Donald murdered Jones; and by hypothesis Donald, far from being (A16.vii) or (A16.ix), is (A16.viii). This conclusion can anyway be validated by supposing that A, instead of suspecting Donald, suspects Smith’s partner Fred; under this new context A believes Fred murdered Smith and B believes he murdered Jones is true if and only if B believes that Jones has been murdered by Fred.

Therefore, since the person B believes murdered Jones must be the person A believes murdered Smith, the anaphoric function states unequivocally that the antecedent of “he” is “he whom A believes murdered Smith”. And the same conclusion holds perfectly with reference to (A16.i), because the difference between (A16.i) and (A16.x), is simply that in the former case A is not in condition to give a precise identity to his suspect; a difference concerning A’s opinion which obviously cannot modify the anaphoric function.

A16.4.1. Exactly as (A16.viii) is the antecedent of the “he” occurring in (A16.i),
he whom B believes murdered Jones is the antecedent of the “he” occurring in (A16.ii).

Realizing that, exactly as
(A16.i*) A believes someone murdered Smith and B believes he whom A believes murdered Smith murdered Jones is the only strict periphrasis of (A16.i),
(A16.ii*) B believes someone murdered Jones and A believes he whom B believes murdered Jones murdered Smith is the only strict periphrasis of (A16.ii), is overcoming the puzzle. In fact the condition assuring the truth of (A16.i*), that is the identity between the person A believes murdered Smith and the person B believes murdered Jones, is perfectly reflected in the condition assuring the truth of (A16.ii*), that is the identity between the person B believes murdered Jones and the person A believes murdered Smith (by hypothesis both of them identify themselves with the person B believes murdered Smith).

A16.4.2. Formally and generally speaking, wherever a
(A16.xii) \( P_x \) occurs within the scope of a B as for instance in
(A16.xiii) \( B_a(P_x) \)
the x we are speaking of is not
(A16.xiv) \( \exists x(P_x) \) but
(A16.xv) \( \exists x[B_a(P_x)] \)
(the individual being P) because the speaker, far from stating that x is P, states that A believes that x is P. The speaker might even dissent from ((A16.xii) and nevertheless he could legitimately state ((A16.xiii). This trivial conclusion can be also supported by comparing
(A16.xvi) \( P_x \& B_a(Q_x) \)
with
(A16.xvii) \( B_a(P_x) \& B_a(Q(\exists x[B_a(P_x)]) \)
and by remarking that, exactly as ((A16.xiv) is the antecedent of the second individual variable occurring in ((A16.xvi), (A16.xv) is the antecedent of the second individual variable occurring in ((A16.xvii).

Therefore the strict periphrases of ((A16.xvii) are
\( B_a(P_x) \& B_a(Q(\exists x/B_a(P_x))) \)
and
\( B_a(Q_x) \& B_a(P(\exists x/B_a(Q_x))) \)
respectively. The root of Edelberg’s untenable claim is mistaking
(A16.xviii) \( B_a(Q_x) \& B_a(P(\exists x/B_a(Q_x))) \)
for ((A16.xvii), that is mistaking A’s and B’s (incompatible) statutes apropos of *he who murdered Jones*.

A16.4.3. The intriguing collateral question
why does the truth of (A16.ii) need a subtle analysis while the truth of (A16.i) is evident?
can be immediately answered; while mistaking (A16.viii) for (A16.ix) is inconsequential on the truth of (A16.i) because A’s and B’s statutes agree apropos of Smith’s murderer, mistaking
(A16.xx) he whom A believes murdered Jones for (A16.xi), that is, symbolically, mistaking (A16.xviii) for (A16.xix), upsets the truth value of (A16.ii) because A’s and B’s statutes disagree apropos of Jones’s murderer. And mistaking (A16.xx) for (A16.xi) is suggested by a hasty reading of (A16.ii) where “… someone murdered Jones … A believes he murdered Smith” actually occurs.
A16.5. Geach’s original puzzle concerns the (seeming) impossibility of theorizing a de re or a de dicto use of pronouns in intentional identity contexts. Since from the informational viewpoint such a distinction is of no moment because the ontological dimension is not necessary for the accomplishment of a semantic course (cannot we properly speak of Polyphemus?), I propose an over-schematized but more direct version of the same puzzle, where the problem concerning the real existence of witches can be completely neglected (we speak of witches exactly as we could speak of Cyclopes).

Example 1. Hob and Nob agree that the witch Hob thinks blighted Bob’s mare is the witch Nob thinks killed Cob’s sow.

Example 2. Nob is unaware of Hob’s existence. Yet both of them know that Peter thinks a pestiferous witch roamed over the hills. Hob and Nob think (independently, of course) their witch is Peter’s one.

The puzzle runs as follows. Undoubtedly

(A16.xxi) Hob thinks a witch blighted Bob’s mare and Nob thinks she killed Cob’s sow is true with reference to both examples; therefore, since, on the grounds of the conclusion achieved in §4 above,

((A16.xxii) the witch Hob thinks blighted Bob’s mare

is the antecedent of the “she” occurring in ((A16.xxi),

(A16.xxiii) Hob thinks a witch blighted Bob’s mare

and Nob thinks the witch Hob thinks blighted Bob’s mare killed Cob’s sow

is obtained from ((A16.xxi) simply by replacing “she” with its antecedent ((A16.xxii)). Yet while ((A16.xxiii) is true with reference to Example 1, it cannot be true with reference to Example 2, where Nob is unaware even of Hob’s existence.

A16.5.1. This reasoning, once more, violates the competence condition. While in Example 1 what the speaker knows, then in particular the identity

(A16.xxiv) the witch Hob thinks blighted Bob’s mare is the witch Nob thinks killed Cob’s sow

is also known by Nob and as such can be used in any inference concerning Nob’s beliefs, in Example 2, by hypothesis (p.2: Nob has no belief at all about Hob or about Bob’s mare) the speaker and Nob have different statutes; in particular (A16.xxiv) is an identity known only by the speaker; therefore it cannot be used in any inference concerning Nob’s beliefs. With reference to Example 2, the only legitimate antecedent for the “she” occurring in ((A16.xxi) is

the witch Peter thinks roamed over the hills

because

(A16.xxv) the witch Peter thinks roamed over the hills is the witch Nob thinks killed Cob’s sow

is the only identity Nob knows. And actually

(A16.xxvi) Hob thinks a witch blighted Bob’s mare

and Nob thinks the witch Peter thinks roamed over the hills killed Cob’s sow

is an unobjectionable inference. In this sense the simple respect of the competence condition prevents us from deriving any puzzle.

In other words. With reference to Example 1, so to write, (A16.xxi) is speaker-true and Nob-true, but with reference to Example 2 (A16.xxi) is only speaker-true.

A16.5.2. The formal approach to Geach’s puzzle follows plainly from the considerations of §A16.4.2. The competence condition is clear: only data (in particular, only identities) occurring within the scope of a “B_{g^*}” can be used for any derivation within the scope of the same “B_{g^*}”. So while

\[ B_{g^*}(x=y) \land B_{g^*}(P_{x}) \supset B_{g^*}(P_{y}) \]

is a misleading rule, and in particular (where \(g^*\) is the speaker)

\[ [(x=y)) \land B_{g^*}(P_{x})] \supset B_{g^*}(P_{y}) \]

is a peculiarly misleading rule,

\[ B_{g^*}(x=y) \land B_{g}(P_{x}) \supset B_{g}(P_{y}) \]

is the right one.

A16.5.3. In order to sketch formally few details of the reasoning proposed in §A16.5 1, a simplification is possible; in fact, once assumed that the individual variables range directly over witches, the omnipresent predicate “to be a witch” can be omitted. Under this assumption

\[ B_{p}[Ex(Bx) \land Ey(Cy)\&(y=x)] \land B_{h}[Ex(Bx) \land Ey(Cy)\&(y=x)] \]

symbolizes Example 1; then, by double substitution of identity,

\[ B_{h}[Ex(Bx \land Cx)] \land B_{h}[Ex(Bx \land Cx)] \]

(the double substitution is correct because “y=x” occurs both in the scope of “B_{p}” and of “B_{h}”, that is because both Hob and Nob are aware of the identity between their witches).

Analogously the symbolization of Example 2 is

\[ B_{h}[Ex(Rz)] \land B_{h}[Ex(Ex(Bx) \land Ez(Rz)\&(x=z))] \land B_{h}[Ex(Ex(Bx) \land Ez(Rz)\&(y=z))] \]

where “B_{h}” symbolizes “Peter thinks that” and “R” is the predicate for the pestiferous witch roaming over the hills. If we look at the question from the speaker’s viewpoint, where
(A16.xxvii) \( \iota z(B_p(Rz)) = \iota x(B_h(Bx)) \)

(the speaker knows that the witch Peter thinks roamed over the hills is the witch Hob thinks blighted Bob’s mare), and where

(A16.xxviii) \( \iota z(B_p(Rz)) = \iota y(B_n(Cy)) \)

(the speaker knows that the witch Peter thinks roamed over the hills is the witch Nob thinks killed Cob’s sow), we can immediately derive

\( \iota y(B_n(Cy)) = \iota x(B_h(Bx)) \)

(the witch Hob thinks blighted Bob’s mare is the witch Nob thinks killed Cob’s sow), therefore we can derive

\( \iota y(B_n(Cy)) = \iota x(B_h(Bx)) \)

violates the competence condition. Briefly; since the identity ((A16.xxvii) cannot occur within the scope of “\( B_n \)”, (Nob knows that his witch is the witch Peter thinks roamed over the hills) but we cannot put ((A16.xxvii) within the scope of “\( B_n \)” (Nob does not know that the witch Peter thinks roamed over the hills is the witch Hob thinks blighted Bob’s mare), therefore the derivation of

\( \iota x(B_h(Bx)) = \iota y(B_n(Cy)) \)

is immediately derivable, yet from such a viewpoint no puzzle arises since we are reasoning under the speaker’s viewpoint, and from such a viewpoint ((A16.xxiii) is perfectly true because, though Nob unaware, his witch is Hob’s witch.

If on the contrary we look at the matter from Nob’s viewpoint, we can put ((A16.xxviii) within the scope of “\( B_n \)” (Nob knows that his witch is the witch Peter thinks roamed over the hills) but we cannot put ((A16.xxviii) within the scope of “\( B_n \)” (Nob does not know that the witch Peter thinks roamed over the hills is the witch Hob thinks blighted Bob’s mare), therefore the derivation of

\( \iota y(B_n(Cy)) = \iota x(B_h(Bx)) \)

violates the competence condition. Briefly; since the identity ((A16.xxvii) cannot occur within the scope of “\( B_n \)”, it cannot be used for a substitution within the scope of such an operator.

In conclusion. From the speaker viewpoint ((A16.xxiii) can be correctly derived, yet from such a viewpoint no puzzle arises because it is true. In order to make ((A16.xxiii) false, we must read it from Nob’ viewpoint, but then it is incorrectly derived. So, once we realize that such a falsity is not born by the illegitimate reading of the pronoun as a pronoun of laziness, but by the violation of the competence condition, Geach’s puzzle vanishes.

A16.6. The root of the impasse concerning Edelberg’s Example 7 (p.18), in spite of the more intricate context, is the same; therefore its solution can be achieved through the above analysis. Shortly his claim (p.17) according to which on its most natural reading Ed(26) is true and on its most natural reading Ed(27) is false is untenable. In fact the only reading under which

Ed(26) A thinks someone murdered the mayor and
B thinks he murdered the commissioner
is true can be the speaker’s reading (A does not know anything about the commissioner and his death; B does not know anything about the mayor and his death). But under the speaker’s reading also

Ed(27) B thinks someone murdered the commissioner
and A thinks he murdered the mayor
is true because all the four following identities

- the person B thinks murdered the commissioner is the person B thinks shot Jones
- the person B thinks shot Jones is the person B thinks shot Smith
- the person B thinks shot Smith is the person A thinks shot Smith
- the person A thinks shot Smith is the person A thinks murdered the mayor

are known by the speaker (they all are explicit assumptions of Example 7) and as such can legitimately be used for substitutions of identity concerning \( k^{sp} \).

A16.6.1. As for the intriguing question
why does Ed(27) need a subtle analysis while Ed(26) does not?
I simply remind the reader of §A16.4.3 above.

A16.7. Also the well known Surprise Text paradox (Binkley 1968; J.H. Halpern, Y.Moses 1986) is born by a violation of the competence condition, yet such a violation concerns the chronological dimension.

I abridge the paradox as follows. A teacher announced to his private pupil Eve that on exactly one of the next two lesson days he would give her a test; but it would be a surprise test in the sense that on the evening before the test she would not know that the test would take place the next day. Eve charges the teacher with incoherence; in fact, she argues
- the test cannot take place the second day for in this case on the evening of the first day she would expect it
- the test cannot take place in the first day for, having already eliminated the second day, she would expect it.

A16.7.1. In order to formalize the situation I assume “\( d_i \)” for “ the test takes place day \( i \)” and

\[ d_1 \Uparrow d_2 \]

for

\[ \neg (d_1 \& \neg d_2) \& (d_1 \& d_2) \]

I recall that “\( \Uparrow \)” is the symbol for the partitive disjunction XOR, conjunction of the inclusive and the exclusive disjunctions).

Let \( k^o \) be Eve’s initial knowledge concerning the possible dates of the test; since

\[ k^o = (d_1 \Uparrow d_2) \]
if we suppose that the test did not take place on the first day,
\[ k' = k \& \sim d_1, \]
is her knowledge at the evening of the day 1; then manifestly the acquirement
\[ \sim d_1 \]
makes \( k' \) and \( k \) two different statutes.

The condition of surprise
\[ S(d_j) \]
is by definition
\[ (A16.xxix) \quad \sim (k' \supset d_j) \]
(where of course \( k' \) is Eve’s statute at the evening before the \( j \)-day).

A16.7.2. The paradoxical flavour of the situation depends on the incompatibility between the conclusion drawn by Eve (no surprise is possible) and the actual component of surprise entailed by the lack of information (\( d_1 \neq d_2 \)) affecting the announcement. But such an incompatibility is born by the ambiguity of ((A16.xxix); that is by the ambiguity concerning the range of the variable "\( j \)." Does the teacher claim that the condition of surprise is only valid at the moment he utters it (\( j=1 \)) or does he claim that it must be valid for the whole period (\( j=1 \) and \( j=2 \))?

In the former case his announcement is true because
\[ \sim ((d_1 \not\supset d_j) \supset d_j) \]
that is
\[ \sim (k \supset d_j) \]
satisfies (A16.xxix); therefore Eve’s argument is illegitimate because the presumed incoherence of the announcement depends on
\[ (A16.xxx) \quad S(d_j) \]
which is not claimed by the teacher.

In the latter case Eve is right; in fact the announcement is incoherent because ((A16.xxx) is actually claimed, and
\[ (d_1 \not\supset d_2 \& \sim d_1) \supset d_2 \]
that is
\[ k' \supset d_2 \]
contradicts ((A16.xxix).

A16.7.3. The extrapolation of the above analysis to \( n>2 \) days is only a matter of notational patience; the easy conclusion is that a precise agreement about
\[ S(d_n) \]
can avoid any impasse. Furthermore the reason is explained why as days go by and the last possible date approaches, the surprise effect decreases (the number of partitioned alternatives is progressively reduced).

A16.8. Another violation of the competence condition involving the chronological dimension is instanced by usual statements like
\[ (A16.xxxi) \quad \text{I believed that Jim was younger than he really was.} \]
Of course ((A16.xxxi) is incoherent if interpreted as
at \( t^0 \) I believed that Jim’s age was less than itself
while the same ((A16.xxxi) is perfectly reasonable if interpreted as
at \( t' \) I have acquired information sufficient to correct
my previous belief about Jim’s age
since the recognized existence of two chronologically different statutes legitimates the existence of incompatible beliefs.

A16.9. The most consequential violation of the competence condition is Frege’s argument inducing him to claim that the content of a proposition cannot be its Bedeutung. Since the identity between Morningstar and Eveningstar does not hold in the statute of an individual who does not know it, the incidental difference in the alethic values of the two respective propositions is not evidence.
CHAPTER 17
REFLEXIVE VARIABLE

17.1. Reflexivity is a manifest case of indexicality and represents an insidious theme that I intend to analyze in detail, owing to its primary importance for the history of logic.

17.2. Let \( L \) be a symbolic language and \( ML \) its metalanguage. In \( L \) "a" and "b" are individual constants (for "Ava" and "Bob"), "x" and "y" are individual variables, "A" and "B" are verbs (for "to admire" and "to blandish"). In \( ML \) "Q", "R", "T" and "Φ" are (syntactical) variables respectively for monadic predicates, dyadic predicates, terms and monadic functors. For instance, then, the \( L \)-translation of

(17.i) she admires him

is

(17.ii) \( A(x,y) \)

(she who? he who?). Analogously

\( A(x,b) \)

is the \( L \)-translation of

(17.iii) she admires Bob

(she who?) and

(17.iv) \( A(a,y) \)

is the \( L \)-translation of

(17.v) Ava admires him

(him who?). Here too (§15.14.1) the discourse is valid until the sentences are considered separately. In fact as soon as they are inserted in a wider context where the pronouns have an opportune antecedent, the situation changes. For instance the correct \( L \)-translation of

(17.vi) if Ava blandishes Bob, then she admires him

is

\( B(a,b) \rightarrow A(a,b) \)

because what (17.vi) tells us is exactly that she is Ava and he (him?) is Bob. Therefore

\( A(a,b) \)

is now the correct \( L \)-translation of the same (17.i) whose previous correct \( L \)-translation was (17.ii).

I insist in emphasizing that the aim of the syntactical approach is to comply with the current ones. The very core of the discourse continues being semantic. If I utter (17.i) when my interlocutor and I are looking at Ava and Bob chatting pleasantly on a desert shore, the conversion of the two variables is performed without any grammatical antecedent because the anaphoric function finds the integrative informational source in the two persons we are looking at.

17.3. The theoretically crucial passage is that while natural languages do possess the possibility to formulate directly reflexive relations, current symbolic languages do not; and though at a first and superficial sight such an impossibility may seem non-reductive, it actually mutilates the expressive power of the languages in question because reflexive predicates singularly considered become unattainable.

Even the well-developed natural languages do not represent a sophisticated linguistic model; wants, ambiguities and redundancies are too many. On the other hand the well-developed natural languages are a highly tested means of communication, and anyhow they represent the best model at our disposal. Of course among well-developed natural languages many expressive differences do exist; yet the common presence of some basic characteristics in all of them (from Port Royal to Leibniz and to the same Chomsky) suggests the presence of some universal requirements dictated by a universal \textit{modus cogitandi}. As far as I know reflexive variables occur in every well-developed natural language; and as such we either arrogantly ignore the result of a linguistic experience ripened in thousands of centuries, or we have to acknowledge the opportunity of their presence in artificial symbolic languages too.

17.4. While we can easily and unproblematically symbolize

(17.vii) Ava admires herself

by

(17.viii) \( A(a,a) \)

as soon as we try to symbolize separately the reflexive predicate

(17.ix) to admire oneself

we realize immediately that the lack of a reflexive variable actually mutilates the expressive power of the same symbolic language. In fact

\( A(\ldots,x) \)
symbolizes to admire the person to specify
and
\[ A(...,a) \]
symbolizes to admire Ava
(just because in (17.viii) “a” occurs both as the subject and as the complement, the same (17.viii) can symbolize (17.vii); but this procedure by iteration, obviously, fails where, as in (17.ix), we deal with a reflexive predicate alone).

In order to overcome this heavy limit I introduce in L a reflexive individual variable “s” (or even, for the sake of generality, a set of reflexive variables “s_1”, “s_2” et cetera) whose conversion rules will be discussed below. For the moment I legitimate its right to exist by remarking that
\[ A(...,s) \]
symbolizes exactly (17.ix). Analogously, for instance,
\[ A(a, s_1) & A(b, s_2) \]
says (roughly) that both Ava and Bob are narcissists (Ava admires herself and Bob admires himself).

17.4.1. With reference to §16.6,
\[ d_{b_{10}} > 500 \]
is the correct symbolization of “peripheral”; but without “s”, which assures the right conversion of the indexical predicate into the contingent absolute ones (“far from Madrid” if the subject is “Cadiz”, “far from Rome” if the subject is “Trieste” et cetera), how could we carry out such a symbolization?

To realize such an impossibility is to realize that the introduction of reflexive variables is a due (and very useful) enrichment of symbolic languages.

17.5. Reflexive and non reflexive (or generic) variables constitute two sorts of individual variables, and the theoretically discriminating factor between them is that they obey different conversion rules.

Here is an easy example. The only difference between

(17.x) When Ava is ill, Eve takes care of her
and
(17.xi) When Ava is ill, Eve takes care of herself

corns the two pronouns: generic in (17.x), reflexive in (17.xi). The respective messages are manifestly different: while (17.x) suggests a charitable Eve, (17.xi) suggests a neurotically egoistic Eve. And this difference follows from the different antecedents of the two pronouns (variables): while “her” stands for “Ava”, “herself” stands for “Eve”. Then we must distinguish between the anaphoric function ruling generic indexical variables, and the anaphoric function ruling reflexive variables. Here too (§16.5.2) we can avoid the detailed theorization of the conversion rules for reflexive variables, since the sentences of our very interest are not affected by any problem of conversion. In fact it is sufficient to agree that where the reflexive variable occurs in the predicate of an atomic sentence, its antecedent is the respective subject.

17.5.1. A subtle question runs as follows. In (17.x) no reflexive pronoun occurs, and the syntactically possible antecedents of “her” are “Ava” and “Eve” (which, furthermore, is the nearest one); why on earth, then, when we read (17.x) do we all understand unequivocally that the right antecedent is “Ava”?

The answer is simple. Since we all know that English possesses reflexive pronouns, we also know that in order to assume Eve as referential antecedent we should use “herself”; then, since in (17.x) a non-reflexive pronoun is used, its referential antecedent is not Eve. In other words, Eve is a no longer available antecedent for the non-reflexive anaphoric function.

17.5.2. Deep searches on the notion of reflexivity are not necessary to understand that, singularly considered, a generic variable (*the individual to specify*) adduces less information than a reflexive one (*the individual we are speaking of*). Here is the simple reason why (I follow Quine’s symbolism in Methods of Logic, §25)
\[ G_{yy} \supset \exists x (G_{xy}) \]
is a valid scheme, while
\[ G_{xy} \supset \exists x (G_{xx}) \]
is not. In other words: since *that same individual* implies *an individual* but not vice versa, the truth of

(17.xii)
\[ G_{yy} \]
is sufficient to assure the truth of

(17.xiii)
\[ \exists x (G_{xy}) \]
but the truth of (17.xiii) is not sufficient to infer the truth of (17.xii). In this sense here too there is no need to adopt integrative conditions (which, moreover, are not supported by theoretical arguments but only by counter-examples, as for instance in the quoted Quine’s work).
17.6. While the examples above concern the direct reflexivity,
concerns an indirect reflexivity. Nevertheless the difference is nearly negligible. In fact, since (17.xiv) can be symbolized in something like
both the direct and the indirect reflexivity enter into the general $ML$-scheme
as soon as the direct reflexivity is conceived as the particular case where $\Phi$ is the null functor of identity. That is: the theoretically essential achievement is not the absence of a functor in
but the necessary presence of the reflexive variable both in (17.xv) and (17.xvii).

17.6.1. Reciprocity too is a case of indirect reflexivity. For instance
means that Ava greeted her greeter and Bob greeted his greeter, then (17.xviii) too enters into the scheme (17.xvi).

17.7. An easy application of the reflexive variable allows us an immediate solution of Kripke’s paradox. Its well known derivation runs as follows. Since
and since
(where “$=_{nec}$” symbolizes the relation of necessary equality), then, particularizing $P$ on the predicate
we get
and finally
a conclusion in manifest contrast with our common sense.

The immediate solution points out that the actual predicate is not (17.xix) but
(actually every individual is necessarily equal to himself or herself or itself, not necessarily equal to $x$): and (17.xx) is an indexical predicate that by $s$-conversion becomes “$=_{nec}y$” when it is ascribed to $y$, just as it becomes “$=_{nec}x$” when it is ascribed to $x$.

17.7.1. The ordinary solution of Kripke’s paradox points out that (17.xix) is not a correct particularization of $P$ because it binds “$x$”. Yet such a solution is affected by a strong theoretical fault since (§15.15) just under the current orthodoxy, the incorrectness of such particularizations is not a theorem, but only an assumption grounded on counter-examples.

17.8. I think that the very root of the misleading approach to logical paradoxes is the current classification of open sentences. In compliance with a famous Skolem’s suggestion (1922), the classifying criterion is the number of different free variables occurring in the open sentence under scrutiny (monadic, dyadic et cetera), without minding the number of occurrences for every free variable. Not to overcharge the analysis, I only consider sentences in model form, once agreed that a monadic sentence is in model form iff
- its subject consists of the simple free variable
- its predicate is either free-variable-free or free-variable-laden
(of course in a monadic sentence whose subject is free-variable-laden, either the predicate is free-variable-free or its variable must be the same of the subject, therefore in this case the sentence is reflexive).

Considering only sentences in model form is not so reductive a simplification as it may seem. For instance,

"Ava’s mother was run over by the eldest sister of his wife" is a monadic sentence (the wife of whom?) whose form is far from the model one; yet it is easy to transform (17.xxi) into
and to ascertain that (17.xxii) is in model form. Anyhow, even if the simplification were highly reductive, it does not forbid me to tackle the logical paradoxes and to give them a general solution.

17.8.1. I provisionally epitomize these conditions in the symbolic scheme

where \( V \) (without inverted commas, I am using the metalinguistic symbol to speak of object expressions) is the \( L \)-variable (as “\( x \)”, “\( y \)” et cetera) constituting the subject of the model sentence, and \( P \) (idem) is a \( L \)-predicate either free-variable-free or free-variable-laden.

17.9. The crucial achievement is that even under these simplifications, Skolem’s criterion is highly unsatisfactory: in fact it does not introduce any distinction between monadic sentences like \((17.xxii)\), whose predicate

\[
\text{is free-variable-free (absolute), and model sentences like}
\]

\[
\text{he is the husband of a lady whose eldest sister ran over Ava’s mother}
\]

\[
\text{ whose predicate}
\]

\[
\text{(17.xxv) being the husband of a lady whose eldest sister ran over his mother}
\]

\[
\text{whose predicate}
\]

\[
\text{(17.xxv) being the husband of a lady whose eldest sister ran over his mother}
\]

\[
\text{is free-variable-laden and more precisely reflexive.}
\]

17.10. Exactly because both \((17.xxiv)\) and \((17.xxv)\) enter into the provisional scheme \((17.xxiii)\), such a scheme does not account for the basic distinction between free-variable-free and free-variable-laden predicates. In order to overcome such a mutilating inadequateness, once assumed “\( C \)” as a metalinguistic symbol over \( L \)-constants, I oppose the scheme

\[
(17.xxv) \quad P_{\Phi(C)}(V)
\]

or indifferently

\[
(17.xxvi) \quad R(V, \Phi(C))
\]

to the scheme

\[
(17.xxvii) \quad P_{\Phi(C)}(V)
\]

or indifferently

\[
(17.xxviii) \quad R(V, \Phi(“s”))
\]

so overcoming the ambiguity of \((17.xxiii)\), where \( P \) may be a free-variable-free as well as a free-variable-laden predicate without compromising the monadic character of the respective sentence.

17.11. Until now I have spoken of sentences, but henceforth I prefer to speak of dilemmas because I think that it should be better and clearer. So, for instance, I will say that while \((17.xxv)\) and \((17.xxvi)\) are schemes of non-reflexive (absolute) dilemmas, \((17.xxvii)\) and \((17.xxiii)\) are schemes of reflexive dilemmas. And incidentally I remark that paradoxical and anti-paradoxical (and para-paradoxical) dilemmas belong to the latter case.
CHAPTER 18
LOGICAL PARADOXES

18.1. Logical paradoxes are a challenge to the human mind. A challenge I presume to have won. Along the lines of the distinction proposed by Ramsey in *The foundations of mathematics*, today there is a tendency to classify as logical only the paradoxes which can be formulated in the language of a formal theory. Thus the father Liar, but also Grelling and so on are classified as epistemological paradoxes. In my opinion the analogies between the two classes are too profound to make momentous such a distinction. I join then Goddard and Johnston where they claim (1983, p.491) that *what is important about the paradoxes is their common feature, not their differences ...in this sense all are logical*). Of course “all” must be interpreted in the nearly obvious acceptation according to which the quantification does not concern absolutely distinct disciplines (as for instance hydrostatics (D’Alembert) or probability (Bertrand). On the other hand I am ready to classify among logical paradoxes results that today are not considered as paradoxes (Schoenfinkel’s reduction), and even results that today are considered supreme achievements (Cantor’s proof, Gödel’s incompleteness theorems).

18.2. I agree with Nelson-Lesniewski’s claim (Sobociński, 1949) according to which a paradox is much more than a contradiction: it is a contradiction derived through a seemingly unexceptionable argument from seemingly unexceptionable premises. Consequently every procedure that forbids the derivation of the paradox without explaining clearly the logical mistake it is born by, represents only a sheer expedient; for instance it is a sheer expedient to banish some intuitively adequate rule of transformation or to mutilate the language in order to forbid the very possibility of formulating the paradox. Since I join Goddard and Johnston also where they write *Any technique which successfully removes the contradiction ... we call a resolution, rather than a solution ..... a solution is a resolution together with a rationale*, I can concisely claim that in order to overcome logical paradoxes what we need is a solution, not a resolution.

18.3. The basic presupposition is that no contradiction can be derived from coherent assumptions through a coherent reasoning. Therefore a derived contradiction is unquestionable evidence that either the assumptions or the reasoning hide some incoherence (which an easy etymologic suggestion induces me to call “pseudologism”). I speak in singular because, once more, I join Goddard and Johnston also in emphasizing the similarity of structure characterizing all logical paradoxes: *this similarity of structure is fundamental. It justifies the intuitive view that piecemeal solutions are not solutions at all, and that there must be a single solution which applies to every paradox.* That is: the general solution of logical paradoxes is the clear explanation of the pseudologism they arise from. In this sense the solution must satisfy three prejudicial conditions, and precisely:

I. To be powerful enough to enclose every known logical paradox (it must kill the central ganglion-cell of the octopus, not simply cut off a single tentacle).
II. Not to be so powerful to interdict argumentative techniques elsewhere useful and legitimate (it cannot kill the octopus by an explosion destroying the whole sea-fauna).
III. To be highly convincing on the ground of a general approach to logic (the denunciation of a solecism, once well understood, cannot be a baffling claim).

If I am not mistaking, the solution I propose satisfies these three prejudicial conditions. Here I expose it both through a formal and an informational approach. Yet there is a third and intuitively enlightening approach that, if Gods are benevolent, I will expose in a book specifically devoted to a representation of semantics.

18.4. First of all I recall
- that a dilemma is either closed (free-variable-free) or open (fee-variable-laden);
- that a closed dilemma is a question liable to two opposite answers;
- that an open dilemma is affected by an intrinsic lack of information forbidding any answer.

Until now I have spoken of (logical) paradoxes under the usual acceptation of the term, but henceforth I will speak of paradoxes to mean also what are usually called “antiparadoxes” (as for instance the Truthteller). My choice is justified by two pieces of evidence:

a) a dilemma whose two opposite solutions are self-confirming is not less puzzling than a dilemma whose two opposite solutions are self-contradictory
b) the same pseudologism leading to paradoxical dilemmas leads to anti-paradoxical dilemmas too, so that the same argument solves all of them.

18.4.1. Since my actual aim is to present the general solution in its most plain version, I do not pursue the widest approach, but the most accessible one. For instance only textual conversions are considered (that is: deictic conversions are neglected).
In fact such a solution can be summarized in the recognition that paradoxical dilemmas are (crypto) defective because, syntactically reasoning, the same free variable to fix occurs free also in the antecedent through which it ought to be fixed.

Indexical dilemmas (more particularly: reflexive dilemmas) play a crucial role in the general solution of logical paradoxes. In fact such a solution can be summarized in the recognition that paradoxical dilemmas are (crypto) defective because, syntactically reasoning, the same free variable to fix occurs free also in the antecedent through which it ought to be fixed.

In other words paradoxical dilemmas are preposterous in the strictly etymologic acceptance of “preposterous” (*before after*) because their previous solution ought to represent the datum on whose grounds the same solution could be achieved. And the hypothetical procedure (if it were... then it ought to be...) by which the various paradoxes are characterized is exactly the consequence of their preposterousness, that is the vain attempt to overcome by a hypothetical postulation an intrinsic lack of information. In this sense Humphries’s distinction between diagonal (constructive) and heterological (non-constructive) procedures concerns a secondary aspect of the matter.

18.5. The standard (extensional) procedure towards paradoxes can be schematized as follows
- to consider a set of individuals \( x \)
- to consider a set of sets \( Y \)
- to define a correspondence \( \eta \) between \( x \) and \( Y \) so that \( Y = \eta x \)
- to partition the pairs \((x,Y)\) on the grounds of a relation \( \phi \) between \( x \) and \( Y \)
- to consider the sets \( A \) and \( Q \) of those \( x \) such that \( \phi \eta x \) and \( \neg \phi \eta x \) respectively
- to select the individuals \( a \) and \( q \) such that \( \eta a = A \) and \( \eta q = Q \)
- to realize that the dilemmas \([a \in \phi \eta a] \) and \([q \in \phi \eta q] \) (or indifferently \([a \not\in \phi \eta a] \) and \([q \not\in \phi \eta q] \)) are defective.

The intensional procedure, once we adequately the notations above to an intensional context, is exactly the same; for instance the set \( A \) becomes the attribute (property) \( A \) and so on.

18.6. Let me apply this scheme to Richard’s paradox (in one of its versions). Though for the sake of concision I define each set through the respective condition of membership, the discourse is even more easily applicable to sets defined through the list of their respective members.

Let \( \rho \) be a correspondence between natural numbers \( x \) and sets of natural numbers \( Y \). Under \( \rho \), say,

\[
\rho(4) = \text{the set of even numbers, that is } y : \exists (z + z = y) \\
\rho(7) = \text{the set of quadratic numbers, that is } y : \exists (z^2 = y) \\
\rho(9) = \text{the set of cubic numbers, that is } y : \exists (z^3 = y) \\
\rho(13) = \text{the set of prime numbers, that is } y : \neg \exists w (w = 1) \land w = y \land w = z = y)
\]

and so on. Of course in order to ascertain whether a number is a member of a set we must ascertain whether such a number satisfies the condition of membership

\[
(x \in \{y : P(y)\}) \leftrightarrow P_x
\]

then for instance, while 6 is a member of \( \rho(4) \) because \( \exists (z+z=6) \), it is not a member of \( \rho(7) \) because \( \neg \exists (z^2 = 6) \) and so on. In particular there are numbers (as 4 and 13) that are members of their set and numbers (as 7 and 9) that are not; it is then possible to define the (Richardian) set

\[
\{y : y \not\in \rho(s)\}
\]

that is the set whose members are the numbers which do not belong to their respective set. In (ii) we could replace “\( s \)” with the usual “\( y \)”, but such a replacement would be the result of a conversion, since the very condition of membership is not

\[
\notin \rho(y) \quad \text{(not belonging to the set assigned to a number to specify) but rather}
\]

\[
\notin \rho(s) \quad \text{(not belonging to the set assigned to itself, a condition obviously satisfied by the examples below). In other words,}
\]

\( \rho(4) \) is the generic predicate and \( \notin \rho(s) \) is simply its particularized conversion where \( y \) is the antecedent of the free reflexive variable. Once agreed that \( r \) is the number such that

\[
\rho(r) = \{y : y \not\in \rho(s)\}
\]

the dilemma

\[
| r \in \{y : y \not\in \rho(s)\} |
\]

is paradoxical since

\[
| r \in \rho(r) \leftrightarrow r \not\in \rho(r) |
\]

follows from (18.i) and (18.iv).
The crucial achievement is realizing that the Richardian set is indexical (§16.8.1), so to call a set whose condition of membership is indexical. While the arithmetical property a number must (not) have in order (not) to be a member of an absolute set is always the same (for instance all the members of $\rho(4)$ are even, all the members of $\rho(7)$ are quadratic, all the members of $\rho(9)$ are cubic, all the members of $\rho(13)$ are prime) the arithmetical property a number must not have in order to be a member of $\rho(r)$ varies with the same number (for instance, while 7 is a member of $\rho(r)$ because it is not quadratic, 9 is a member of $\rho(r)$ because it is not cubic and so on). From a severely formal viewpoint I could also say: while
\[
\exists(z+z=...) \\
\forall(z'=...) \\
\exists(z'^2=...) \\
\sim \exists z'(y=1) \& (y=x) \& yz=...
\]
are respectively the concatenations of primitive or previously defined symbols on whose grounds we form respectively $\rho(4)$, $\rho(7)$, $\rho(9)$ and $\rho(13)$, no analogous concatenation for $\rho(r)$ is possible.

18.6.1. Indeed if we should refer the Richardianity to ordered couples (§6.6), we could by-pass the indexicality; this heterodox approach will be analysed in §10 below, apropos of Grelling; here I insist on the orthodox approach. Under it, since the condition of membership depends on a relation (of non-membership) between any specific number and a set of numbers, such a set must be previously and perfectly known in order to answer any specific dilemma; otherwise our answer may be forbidden by a lack of information. Therefore where the condition of membership depends on a relation (of non(membership) between any specific number and a set of numbers, such a set must be previously and perfectly known in order to answer any specific dilemma, as this lack of information affects only two specific dilemmas: the paradoxical and the anti(paradoxical ones (in fact the argument holds identically for the anti-Richardian set, that is the set of numbers which are members of their respective set). From the formal viewpoint, the presence of the free reflexive variable, is not at all influenced by replacing “$\rho^r$” with “$\rho^s$” in (iv). Just because of this presence, while it is correct, for instance, assuming “$E$” to name $\rho(4)$ or “$C$” to name $\rho(9)$, assuming “$R$” to name $\rho(r)$ would be a misleading step because in a definition like
\[
\rho(x) \iff \rho(y)
\]
the free variable occurring in the definiens does not occur in the definiendum. And as soon as we recognize that the correct formulation must be something like
\[
\rho(x) \iff \rho(y)
\]
the paradox vanishes. In fact, for instance, while $\rho(9)$ is $\rho_E$, and $\rho(9)$ is $\rho_C$, $\rho(9)$ is $\rho_R(\rho(9))$ and in “$\rho_R(\rho(9))$” the free (reflexive) variable continues to occur. Let me insist on this central point. The dilemma
\[
4 \in \rho_R(\rho(9))?
\]
can be answered without any appeal to a hypothetical procedure (if 4 were …) because the conversion of the reflexive variable into
\[
4 \in \rho_E(4)?
\]
that is into
\[
4 \in \rho_C
\]
is effective (E is the previously and perfectly determined set of even numbers and the arithmetical structure of the number 4 (its evenness) is sufficient to entail that it is a member of $E$). On the other hand
\[
r \in \rho_R(\rho(9))?
\]
where the same reflexive variable continues to occur free, and the appeal to a hypothetical procedure (if $r$ should belong …) is only the already mentioned vain attempt to escape an intrinsic lack of information; $r$ can neither belong to the Richardian set nor to its complementary because these sets are defined in such a way that their condition of membership is defective in exactly two particularizations (such sets are locally fuzzy, to follow a terminology Fine ought to appreciate).

Therefore I partially agree with Goedel’s intuition where he writes (1944, p.150): “it might even turn out that it is possible to assume every concept to be significant everywhere except for certain “singular points” or “limiting points”, so that the paradoxes would appear as something analogous to dividing by zero. What I cannot accept is his excess of generalization (every concept) because only where a pretence of auto-conversion is involved we meet such singular points. Trivially: if the barber comes from another village, no paradox arises.

18.6.2. Russell’s paradox, whose classical extensional version can be sketched as follows
\[
x \in y \iff x \notin x
\]
\[
y \in y \iff y \notin y
\]
is nothing but a simplified version of the above scheme and enters into the same solution (y is an indexical set whose intension is $\rho(y)$). In other words. The formally correct definition
\[
x \in c_i \iff x \notin x
\]
(that is: $x \in C \rightarrow x \notin C$)

overcomes any impasse, since

$C_i \in C \rightarrow C_i \notin C$

(that is: $c_i \in C \rightarrow c_i \notin C$

far from being contradictory, expresses an unquestionable truth ("c" is a sort of negation).

18.7. I dwell on a collateral but perhaps intriguing consideration. Once some desperate ad hoc proposals (such as banning tout court reflexive predicates from the language) are rejected, and once some marginal differences between Poincaré’s and Russell’s positions are neglected, impredicativity is the only general solution of logical paradoxes till today suggested. According to it, any entity defined by an expression which contains a bound variable must be excluded from the values of this variable. Unfortunately such a solution is too expensive. This vicious circle principle ... is in danger of mutilating rather than purifying mathematics (Beth 1959, p.499). And precisely with the aim of overcoming this heavy difficulty a distinction has been introduced (Hintikka 1956, p. 244) between an innocent and an insidious impredicativity. According to Beth’s same examples,

$$\forall y ((\exists x = y \& \forall x = y \& \exists z (y = x \& z = y))$$

(which defines the unitary element of a group) and

$$\forall y (\exists z (z \in m \& z > y) \leftrightarrow (y < x))$$

(which defines the least superior border of a set $m$ of natural numbers) and

$$\forall s (x \in s) \rightarrow \forall m ((1 \in m \& (y \in m \rightarrow (y+1) \in m)) \rightarrow x \in m)$$

(which defines the set $n$ of natural numbers) are all innocent (and theoretically precious) impredicative definitions.

On the other hand

(18.v)

$$\forall s (x \in s) \leftrightarrow \neg (x \in s)$$

(which defines the set of sets which are not members of themselves), is an insidious impredicative definition.

But from our viewpoint it is easy to remark that only (18.v), once reformulated through the help of a reflexive variable in order to underline the crucial passage, would involve an auto-conversion. This is the deep reason why (Hintikka, ibidem) paradoxes cannot be formulated in the terms of the 'innocent' impredicativities.

Other proposals (as Hintikka’s strongly exclusive interpretation of quantifiers) succeed in forbidding such auto-conversion but (at least in my opinion) fail in explaining why and where such a strongly exclusive interpretation is necessary.

18.7.1. If we intend to preserve the term “impredicativism” to mean the pseudologism the logical paradoxes are born by,

No indexical free variable can be converted by a free-variable-laden antecedent

is the very rule that bans any impredicativism. But indeed, more than a specific addictive rule, it is simply an obvious requirement telling us that if we violate it, we get an expression where a free variable continues to occur, and treating a free-variable-laden expression as a free-variable-free one is unquestionably incoherent.

In this sense the informational approach does not require any addictive prescription besides the respect of coherence, that is, in our case, not defining an indexical quantity and treating it, although implicitly, as if it were absolute. Neither a paradoxical dilemma is self-contradictory nor an anti-paradoxical dilemma is self-legitimating. Both of them are (crypto)defective because both of them hide a lack of information.

18.8. Since I do not know Grelling’s paradox in his original version, I quote the Britannica (whose terminological choices will be adopted in the following discussion too).

Let us classify the adjectives of the English language as to whether they are self-applicable or non-self-applicable. An adjective is self applicable if it has the property it expresses; e.g. the adjective “short” is self-applicable since it is a short word, but “long” is non-self-applicable since it is not a long word. Every adjective is either self-applicable or non-self-applicable, and cannot of course be both. Which is the case for the adjective “non-self-applicable”? Suppose that it is self-applicable. Then it has the property it expresses, i.e. it is non-self-applicable, contrary to our supposition. Thence it is non-self-applicable. This means that it does not have the property it expresses, the property on non-self-applicability. But this is just another way of saying that it is not non-self-applicable. We have again arrived at two contradictory results.

In order to formalize the solution in the simplest way I agree upon the following symbology:

- "x", "y" ...variables ranging over adjectives
- "b", "m", "f" and "u" “blepharospastic”, “monosyllabic”, “fresh” and “useless” respectively
- "a" and “q” self-applicable” and “non-self-applicable” respectively
- X, Y, B, M et cetera properties respectively adduced by “x”, “y”, “b”. “m” et cetera.
I underline that the use of individual variables and individual constants as “x”, “b” et cetera for adjectives is not at all an abusive license, owing to the metalinguistic viewpoint from which we observe them. That is: L-adjectives are named by ML-substantives (§2.5: substantivizing effect of quotation marks). As soon as we (provisionally) agree to define the self-applicability and the non-self applicability by

\[
\begin{align*}
(18.\text{vi}) & \quad A(x) \leftrightarrow X(x) \\
(18.\text{vii}) & \quad Q(x) \leftrightarrow \neg X(x)
\end{align*}
\]
derive the contradictory

\[
(18.\text{viii}) \quad Q(q) \leftrightarrow \neg Q(q)
\]
by particularization of (18.vii) on q (for the sake of concision henceforth I set aside (18.vi) and reason only on (18.vii)).

Though here too the solution depends on the logical impossibility of promoting an indexicality by an auto-conversion, I enter into details, reminding the reader of the detailed example proposed in §16.6.

18.9. Let (x, Y) be an ordered couple where the adjective x and the property Y are casually joined. By definition a couple is concordant (C) iff Y(x) and discordant (D) iff \(\neg(Y(x))\).

Indeed a more articulate analysis (§9.5.1) ought to distinguish between properly discordant couples as for instance (b, M) where the “~” occurring in “\(\neg M(b)\)” means an oppostive negation, and improper (or improperly discordant) couples as for instance (m, B) where the “~”occurring in \(\neg B(m)\)” means an exclusive negation. Yet I am pleased with the simple opposition between discordant and discordant couples because
- the quoted version of the paradox complies with such a simplified assumption (every adjective is either self-applicable or non-self-applicable)
- the same assumption is the usual one (cf. for instance Church in D.D. Runes The Dictionary of Philosophy under “(Logical) Paradoxes”)
- the argument can be easily extrapolated to a further distinction between proper and improper discordance.

Then

\[
(18.\text{ix}) \quad C(x, Y) \leftrightarrow Y(x)
\]
and
\[
(18.\text{x}) \quad D(x, Y) \leftrightarrow \neg Y(x)
\]
symbolize the above definitions (here too for the sake of concision I set aside (18.ix)). Just as (18.x) defines a property pertaining to ordered couples

\[
(18.\text{xii}) \quad D_M(x) \leftrightarrow \neg Y(x)
\]
defines the Y-discordance, a (relational) property that, obviously, pertains to adjectives.

In order to answer any specific dilemma concerning the Y-discordance of a given adjective it is sufficient to particularize (18.xi). For instance, to ascertain whether b is \(M\)-discordant we proceed as follows:

\[
D_M(b) \quad \neg \neg M(b)
\]
and since we know what “monosyllabic” means and how many syllables “blepharospastic” is formed by, we can conclude

\[
\neg(M(b)) \quad D_M(b)
\]
that is the M-discordance of the adjective under scrutiny. Of course should we start from

\[
C_M(b) \quad \neg C_M(b)
\]
the conclusion would be the same (that is \(\neg C_M(b)\)) as the dilemma is the same.

18.9.1. Since a correspondence \(\eta\) exists between an adjective and the property it means (\(\eta = \sigma\))

\[
X = \eta X
\]
therefore the assumption

\[
(18.\text{xii}) \quad Y = X
\]
transforms the discordance into the non-self-applicability; in fact, under (18.xii) the second member of any ordered couple is just the property expressed by the respective first members. Nevertheless the ‘binary nature’ of every couple is not compromised by the possibility of ‘computing’ its second member in function of its first. Though it pertains to adjectives, the (non)self-applicability continues depending on the expressed property, too. For instance if we interpret “fresh” as a synonymous of “new”, “fresh” is non-self-applicable for it is an old adjective, but if we interpret “fresh” as a synonym of “active”, it is self-applicable.

In other words. Exactly as “he” is a precise and constant indexical substantive which, in different contexts, is converted into different absolute substantives (as “Tom”, “Bob” and so on), “non-self-applicable” is a precise and constant indexical adjective which, in different contexts, is converted into different absolute adjectives (as “polysyllabic”, “useful” and so on). In this sense, just as we say that “he” is a pronoun (a pro-noun), we can say that “non-self-applicable” is a proadjective (a pro-adjective). And in fact where ascribed to “monosyllabic” the non-self-applicability is the polysyllabicity (“non-self-applicable” is converted into “polysyllabic”), whereas ascribed to “useless” the non-self-applicability is the usefulness (“non-self-applicable” is converted into “useful”) and so on. Therefore the presence of a free variable in its symbolization is a formal must, exactly in order not to mistake the typographical invariance of the adjective for the invariance of the added property. Informally, such a tramp is evident in the version
replacing “non-self-applicable” and “self-applicable” with “heterologic” and “homologic”, because, contrary to “self”, “hetero” and “homo” tend to conceal their role of variables.

This conclusion is formally confirmed by a comparison between (18.vii) and

\[ D_x(x) \iff \sim (X(x)) \]

drawn from (18.x) through (18.xii). Realizing that “Q” is an elliptically incorrect notation for “Qx” is realizing the impossibility to obtain an effective auto-conversion, since manifestly in “Qx(x)” the free variable continues occurring free. Furthermore it is also realizing why, on the contrary,

(18.xiii) is “blepharospastic” (non)self-applicable?
or

(18.xiv) is “monosyllabic” (non)self-applicable?

And so on are effective dilemmas (no free variable survives to the conversion of “Qx” into “Qx” or into “Qx” and so on). Finally it is also realizing why Grelling’s dilemma is crypto-defective; because all the other analogous dilemmas, as (18.xiii), (18.xiv) and so on are effective (non-preposterous), that is because, save two exceptions (two singular points), in a scheme like

is …… (non)self applicable?

replacing the dots with the name of an adjective bears a non-defective dilemma.

18.9.2. Another strong argument supporting the indexicality of (non-)self-applicability runs as follows. The (im)properness of any proposition obtained by the attribution of an absolute property to an adjective does not depend on the adjective. For instance, since

“useless” is monosyllabic

is (false but) proper, we can substitute “useless” with whatever other adjective without losing the properness of the resulting proposition (which for instance is true (then proper) if the substitutor is “fresh”, false yet anyhow proper if the substitutor is “blepharospastic” et cetera). On the other hand, though

“useless” is non-self-applicable

is another (false but) proper proposition, if the substitutor is “blepharospastic” we get an improper proposition and if the substitutor is “fresh” we get a proper proposition which will be true or false in compliance with the acceptation of “fresh”. This deep discrepancy depends just on the fact that the meaning of “non-self-applicable” (the property it expresses) varies with the adjective it is attributed to; such a deep discrepancy, then, depends on the deep discrepancy between absolute (invariant) and indexical (variant) properties.

18.10. If we modify the original version of Grelling’s paradox by making Q a property pertaining to ordered couples as D, that is if we assume by definition

(18.xv) \[ Q(x,X) \iff \sim (X(x)) \]

the same Q is no longer an indexical property ((18.xv) is a formally correct formula as (18.x)). Yet no contradiction can be derived from (18.xv); in fact the particularization of x on q (therefore the particularization of X on Q) leads to

(18.xvi) \[ Q(q,Q) \iff \sim (Q(q)) \]

and (18.xvi), far from being contradictory, allows us to deepen the analysis by punctuating the two possible readings of “∼”. In the oppositive reading, since \( Q \) pertains to couples, \( Q(q) \) is improper, and the particularization is then illegitimate. In the exclusive reading, precisely because \( Q(q) \) is improper, the second member of (18.xvi) is true. This means that, under this reading,

\[ Q(q,Q) \]

is true (the couple is non-self-applicable); a conclusion which agrees perfectly with our intuition, for actually \( Q \) (by definition a property pertaining to couples) cannot be properly ascribed to an adjective, therefore \( (q,Q) \) is a non-self-applicable couple.

18.10.1. A willing reader might be tempted to revive the paradox by a definition like

(18.xvii) \[ Q(x,X) \iff \sim (x(x,X)) \]

where actually both variables of the definiens occur also in the definiendum. A misleading temptation, because, contrary to (18.xvii), in (18.xvii) both the definiens and the definiendum have the same argument (that is \( (x,X) \)); then, put shortly, (18.xvii) states an equivalence between \( Q \) and \( \sim X \); but of course, as \( X \) is an indexical property which varies with \( x \), its indexicality entails that “\( Q \)” is an incorrect (and misleading) notation. In other words, (18.xvii) falls into the same formal omission affecting (18.vii).

18.11. The Liar (the Truthteller) too is born by the preposterousness of the paradoxical dilemma. Its peculiarity depends only on the different status of the objects the reflexive variable ranges over; while Richard refers to numbers, Russell to sets and Grelling to adjectives, Liar and Truthteller refer to sentences, that is to objects whose syntactical status is the same as that of the (paradoxical) dilemma under scrutiny. Indeed it would be better to reason about propositions, yet in order to adequate the following discourse to the current terminology I accept not only to reason about sentences but also to speak of false and true sentences though I ought to speak of fallacious and veracious
sentences (the only caution is the use of “FL” and “VR” instead of “F” and “T” respectively). Anyhow a stricter approach is sketched in §18.11.7.

The canonical formulation of the Liar, concisely, runs as follows:

\[(18.\text{xviii}) \quad \text{this sentence is false}\]

is self-contradictory because if it were true, it would state its falsity, so it would be false, and vice versa.

Reciprocally (Truthtellier)

\[(18.\text{xix}) \quad \text{this sentence is true}\]

is anyhow self confirming because et cetera.

Once recalled that, according to Tarski’s approach,

\[(18.\text{xix}) \quad \text{VR}(\text{“Y(x)”}) \iff Y(x)\]

and

\[(18.\text{xx}) \quad \text{FL}(\text{“Y(x)”}) \iff \neg Y(x)\]

are the conditions for truth and falsity, I organize the solution through five progressive examples.

### 18.11.1. First example (standard context). The assignation of an alethic value to (18.xxii)

Ava is blonde \((B(a))\)

that is the solution of the respective dilemma \((B(a))\)?

runs as follows:

\[\text{VR}(\text{“B(a)”})?\]

\[\text{VR}(\text{“B(a)”}) \iff B(a)\]

\[B(a)?\]

or equivalently

\[\text{FL}(\text{“B(a)”})?\]

\[\text{FL}(\text{“B(a)”}) \iff \neg B(a)\]

\[\neg B(a)?\]

(the two opposite questions are equivalent because, obviously, answering one of them is answering the other, that is because their common dilemma depends on the same collative datum, just represented by the piece of information concerning Ava’s hair). And since Ava is raven haired,

\[(18.\text{xxii}) \quad \neg B(a)!\]

is the acquirement we draw from the sight of her hair, so the same (18.xxii) allows us to conclude that \(\neg \text{VR}(\text{“B(a)”})\) or equivalently that \(\text{FL}(\text{“B(a)”})\).

### 18.11.2. Second example (metalinguistic context). The assignation of an alethic value to (18.xxiii)

“Ava is blonde” is (printed with) green (ink) \((G(\text{“B(a)”}))\)

runs analogously as follows:

\[\text{VR}(\text{“G(“B(a)”)”})?\]

\[\text{VR}(\text{“G(“B(a)”)”}) \iff G(\text{“B(a)”})\]

\[G(\text{“B(a)”})?\]

and the collative datum (drawn from the sight of the object sentence (18.xxiii) speaks of) is

\[(18.\text{xxiv}) \quad \neg G(\text{“B(a)”})\]

(the object sentence (18.xxiii) speaks of is not printed with green ink) therefore (18.xxiv) allows us to conclude that \(\neg \text{VR}(G(\text{“B(a)”}))\) or equivalently that \(\text{FL}(G(\text{“B(a)”}))\).

### 18.11.3. Third example (auto(referential context). The assignation of an alethic value to (18.xxv)

this sentence is green \((G(s))\)

runs analogously as follows:

\[\text{VR}(\text{“G(s)”})?\]

\[\text{VR}(\text{“G(s)”}) \iff G(s)\]

\[G(s)?\]

that is, by conversion of the reflexive variable (whose antecedent is the whole (18.xxv))

\[(18.\text{xxvi}) \quad G(\text{“G(s)”})?\]

and the collative datum (drawn from the sight of the sentence (18.xxv) speaks of, that is the same (18.xxv)), is

\[(18.\text{xxvii}) \quad \neg G(\text{“G(s)”})\]

(the sentence (18.xxv) is not printed with green ink) therefore (18.xxvii) allows us to conclude that \(\neg \text{VR}(G(\text{“G(s)”}))\). Of course the mentioned conversion is effective because what occurs in (18.xxvi) is not a free (reflexive) variable, but the name of a sentence where the same variable occurs, that is because the same variable is mentioned, not used, and obviously the name of a variable is a constant exactly as the name of a constant); furthermore the just ascertained possibility of solving (18.xxvi) is the best evidence that it is a free-variable-free dilemma.
Incidentally. Though (18.xxv) is unquestionably auto-referential, the alethic procedure did not present any difficulty. The proposal of overcoming the Liar by forbidding auto-referential sentences is then, at the very least, too severe an intervention (I recall the condition IInd of §18.3).

18.11.4. Fourth example (alethic context). The assignation of an alethic value to

(18.xxvii)  “this sentence is green” is true  

runs analogously:

\[ VR(“G(s)”)? \]
\[ VR(“VR(“G(s)”)”)? \]
\[ VR(“G(s)”)? \]

the collative datum is then represented by the result of a collation. In order to achieve this datum the context must allow us to perform such a collation, and in order to perform such a collation we must previously know both corredata, that is, the proposition added by the object sentence within quotation marks in (18.xxvii) and the cognition drawn from the sight of the ink used to print the same object sentence; since we know both of them, we can actually achieve the result, and since it is anti-collative (since our statute acquires the (meta)cognition that the object proposition is rejected by the object cognition), the collative (meta)datum

\[ VR(“G(s)”)? \]

allows us to conclude that

\[ FL(“VR(“G(s)”)”)? \]

(actually it is false that the object sentence (18.xxvii) speaks of is true).

18.11.5. The fifth example (auto-referential alethic context) is the Truthteller. The assignation of an alethic value to

(18.xxix)  “this sentence is true”  

runs analogously:

\[ VR(“s”)? \]
\[ VR(“VR(“s”)”)? \]
\[ VR(“s”)? \]

and here too, since (18.xxix) concerns an alethic predicate just as (18.xxvii), the collative (meta)datum ought to be represented by the result of a collation. But here the context does not allow us to perform such a collation because we cannot know the second corredatum; in fact it ought to be represented by the result of the same collation under scrutiny (something like Baron Münchhausen pulling himself out of a swamp by his own hair).

In other words. While

(18.xxx)

is an effective dilemma because the typographical aspect of the object sentence allows us to ascertain its (non)greenness (allows us to know the homologous cognition), contrary to (18.xxx)

(18.xxx)

that is, by conversion,

\[ VR(“G(s)”)? \]

is a defective (preposterous) dilemma because the intrinsically relational nature of alethic predicates forbids us to ascertain them on the only ground of the sentence (of the proposition) under scrutiny.

Reciprocally (yet identically), if we start from (18.xxvii), we get

\[ FL(“FL(“s”)”)? \]
\[ FL(“FL(“s”)”)? \]
\[ FL(“s”)? \]

therefore here too the collative datum ought to be represented by the solution of the dilemma under scrutiny.

Both Liar and Truthteller are affected by the same and congenital lack of information.

18.11.6. The ‘double face’ version of the Liar (“the next sentence is true” where the next sentence is “the preceding sentence is false”) is immediately reducible to the classical version. In fact the sentence stating that its next sentence is true, states that the sentence preceding its next sentence is false, then it states its own falsity. Thus the preposterousness is fully maintained.

18.11.7. In order to show how preposterousness can be reduced to an attempt of auto-conversion, let me submit Tarski’s condition to a deeper analysis.

Alethics is an intrinsically relational doctrine because, roughly speaking, it concerns some relations between sentences and facts (better: between propositions and cognitions). A (declarative) sentence \( x \) singles out a proposition \( \alpha \), and a proposition singles out a homologous cognition \( \psi \alpha \) verifying or falsifying it (strictly: a proposition singles out a \( k \)-cognition \( k- \)verifying or \( k- \)falsifying it); therefore two opposite propositions have the same homologous cognition. Just as the above approach (which adopts the canonical viewpoint according to which alethic predicates pertain to sentences) leads to homologous couples \( (\alpha \psi \alpha) \), the (correct) viewpoint (according to which alethic predicates pertain to propositions) would lead directly to couples \( (\psi \psi) \). Anyhow a basic distinction opposes the concordant couples \( C \) defined through
(18.xxxi) \[ C(\sigma x, \psi \sigma x) \leftrightarrow (\sigma x = \psi x) \]
(that is: \(C(\sigma x, \psi \sigma x) \leftrightarrow \sigma x = \psi x)\)
and discordant couples \(D) defined through
(18.xxxii) \[ D(\sigma x, \psi \sigma x) \leftrightarrow \neg (\sigma x = \psi \sigma x) \]
(henceforth, for the sake of concision, I neglect (18.xxxii).

To replace (18.xxxi) with
(18.xxxiii) \[ C(\sigma x) \leftrightarrow (\sigma x = \psi x) \]
would be a formal and substantial abuse. In fact
- formally, in (18.xxxiii) “\(C\)” becomes an abbreviation of “\(= \psi x\)”, then a free reflexive variable disappears;
- substantially, (18.xxxiii) would make the concordance between a proposition and its homologous cognition a property
of the same proposition (which obviously is not, since the homologous cognition is an independent and determinative
datum).

Indeed, once more, what we can correctly derive from (18.xxxi) is
(18.xxxiv) \[ C_{\psi}(\sigma x) \leftrightarrow (\sigma x = \psi x) \]
where the indexicality of the predicate “concordant with its homologous” is correctly witnessed by “\(C_{\psi}\)”
(unquestionably if a homologous couple is concordant, the proposition is concordant with its homologous cognition).
But (18.xxxi) and (18.xxxiv) show that the context is strictly analogous to Richard’s and Grelling’s ones: as \(\sigma x = \psi x\)
is the condition of truth, the indexical predicate \(C_{\psi}\) is nothing but the predicate of truth, and any attempt of auto-
conversion for the free reflexive variable is devoted to an even formal failure.

18.11.8. Let me resume by particularizing the “\(Y(s)\)” of (18.xix) on some already proposed sentences:

<table>
<thead>
<tr>
<th>“B(a)”</th>
<th>“G(s)”</th>
<th>“VR(“G(s)”)”</th>
<th>“VR(“G(s)”)”</th>
</tr>
</thead>
<tbody>
<tr>
<td>(VR(“B(a)”)? )(Tarski’s condition)</td>
<td>(VR(“G(s)”)? ) by conversion (G(“G(s)”)? )</td>
<td>(VR(“VR(“G(s)”)”)? )</td>
<td>(VR(“VR(“G(s)”)”)? )</td>
</tr>
<tr>
<td>(VR(“B(a)”)? )</td>
<td>(VR(“G(s)”)? )</td>
<td>(VR(“VR(“G(s)”)”)? )</td>
<td>(VR(“VR(“G(s)”)”)? )</td>
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The correct conclusion is once more the same: neither the Liar dilemma is self-contradictory nor is the
Truth teller anyhow self-confirming; their congenital lack of information forbids them to be true or false.

In two words. The alethic value of a proposition does not depend only on the same proposition, since it concerns
a link between such a proposition and its homologous cognition. Therefore an alethic predicate can be properly ascribed
to a proposition iff the homologous cognition is non-preposterously attainable.

18.12. Such a conclusion, generally speaking, shows that to ground an apagoge (a \textit{reductio ad absurdum})
on a defective dilemma is a totally spurious argument. In fact if
\[ |h| \]
is a defective dilemma, deriving
\[ T_l(\neg h) \]
from
\[ \neg T_l(h) \]
is a quite abusive inference because, just owing to the defectivity of the dilemma, neither \(h\) nor \(\neg h\) can be \(l\)-true.

18.12.1. The clear and sound assimilation of the above evidence entails that Cantor’s celebrated proof is not at
all a proof. His argument can be sketched as follows. Let \(m\) be a numerable set and \(P_m\) its power set, that is the set of its
subsets. A one to one correspondence \(\kappa\) between the members \(x\) of \(m\) and the members \(y\) of \(P_m\) cannot exist. In fact
should it exist, then a \(b\) and a \(c\) would exist such that
\[ \kappa b = c = \{x: x \notin \kappa(x)\} \]
and the dilemma
\[ b \in c? \]
would lead to a contradiction.
The analogies with Richard’s paradox are more than evident (c is an indexical set and an indexical dilemma becomes decidable only after its conversion into an absolute one et cetera). The preposterousness too is manifest: in order to decide whether b does belong to c (so getting an effective conversion of the free reflexive variable) we should previously know whether b does or does not belong to c. So, once more, we have to remark that the worst logical puzzles ensue from the misrecognized presence of a free reflexive variable.

18.12.1.1.  This confutation, of course, does not show that Cantor’s theorem is misleading; it only shows that his claim is a simple proposal, thus offering a theoretical support to the opinion of some scholars who reject Cantor’s whole theory of transfinites on mere humoral bases.

18.12.2.  A similar situation concerns Cantor’s lemma of diagonalization (Shoenfield 1967, §6.8). It is nothing but the conclusion that no indexical predicate is absolute. What is wrong in a definition like
\[ Q(x) \leftrightarrow \neg P_x(x) \]
that is like
\( (18.xxxv) \)
\[ Q(x) \leftrightarrow \neg P_x(x) \]
is the abusive elimination in “Q” of the free variable occurring in the predicate of the definiens. And as soon as we realize that the formally correct definition ought to be something like
\( (18.xxxvi) \)
\[ Q_x(x) \leftrightarrow \neg P_x(x) \]
we realize that the only (and highly questionable) way to defend (18.xxxv) is to agree that an “s” is elliptically included in “Q” (as for instance a “self” is elliptically included in “heterologic”); but under this agreement “Q” becomes a predicative variable which, obviously, cannot be identified with any absolute predicate “P_b” (with reference to the example of §16.6, b would be the exact homologous of Utopolis). In other words, if we start from the correct (18.xxxvi), no conversion can lead to
\[ Q_x(x) \leftrightarrow \neg P_x(x) \]
because the substitution of “b” to “s” must involve the reflexive variable at the first member, too; the right substitution leads to
\[ Q_x(x) \leftrightarrow \neg P_x(x) \]
where \( Q_b \) is precisely the absolute predicate corresponding to that conversion of the indexical one. And as far as I know, no contradiction is derivable from (18.xxxvi).

18.13.  Another simple application of the distinction between indexical and absolute predicates concerns Thomson’s Little Theorem (Butler 1962, p.94). It claims that, if \( \rho \) is a relation defined on a set m, no member \( x \) of m can be in \( \rho \)-relation with all the \( m \)-members \( y \) which are not in \( \rho \)-relation with themselves. But once we symbolize the theorem in
\[ \neg \exists x (x \in m \& \forall y (y \in m \& \rho(x,y) \leftrightarrow \neg \rho(y,y))) \]
\[ \rho(y,q) \leftrightarrow \neg \rho(y,s) \]
is immediately evident that supposing the existence of a q such that
\[ \rho(y,q) \leftrightarrow \neg \rho(y,s) \]
is identifying the indexical predicate
\[ \neg \rho(s) \]
with an absolute predicate
\[ \rho(q) \]
(here too I could evoke Utopolis). And here too I could underline that the occurrence of “~” is of no theoretical moment, because also
\[ \rho(s) \leftrightarrow \rho(q) \]
is an illegitimate assumption (a Thomson’s Little Counter-theorem is not less valid, although its violation does not lead to any direct formal contradiction).

\( (18.xxxvii) \)
\[ \exists x (x \in m \& \forall y (y \in m \& \Phi(x,y) \leftrightarrow \Phi(y,x))) \]
is misleading if the only condition for the particularizations of \( \Phi_k \)
(just in order to avoid that “y” be abusively bound) is that the same “y” cannot occur free in \( \Phi \) (without inverted commas, I am speaking of an object predicative expression, that is of the concatenation of object symbols named by “\( \Phi \)”). In fact under that only condition no distinction is possible between absolute predicates, where no free variable occurs, and indexical (reflexive) predicates where a “s”, (that is a “s”) occurs. But (18.xxxvii) is exactly the axiomatic scheme of abstraction the current set theories are grounded on. May it be a mere casualty that all the formal contortions the set theorists are constrained to adopt with a view to avoiding contradictions are born by indexical predicates, or better by the mentioned lacking distinction?
18.15. Hoping that a surrealistic joke is welcome (or at least tolerated) I take advantage of a defective apagoge to prove what I modestly call “Gandolfi’s Great Theorem”: it simply states that our universe is incoherent. Here is the proof.

Let me consider the ordered couples of natural numbers. On the one hand, as the simple tabulation

\[
\begin{array}{cccc}
(0,0) & (0,1) & (0,2) & (0,3) & (0,4) \\
(1,0) & (1,1) & (1,2) & (1,3) & (....) \\
(2,0) & (2,1) & (2,2) & (....) \\
(3,0) & (3,1) & (....) \\
\end{array}
\]

is diagonally exhaustible (firstly the only couple whose sum is zero, secondly the two couples whose sum is 1, thirdly the three couples whose sum is 2 and so on), such couples must be numerable.

On the other hand, since they are numerable, we can establish a one to one correspondence \( \theta \) between them and the binary numerical relations definable in English, which surely are numerable. Thus, as the members of an ordered couple are linked by the Gandolfian relation \( GN \) iff they are not linked by the relation \( \theta \) corresponding to the same couple is one of such definitions, there is a couple \( \langle x^0, y^0 \rangle \) such that \( GN = \theta(x^0, y^0) \). And as soon as we consider (18.xxxviii)

\[
GN(x^0, y^0)
\]

we fall into a contradiction.

Here too the impasse follows from a \textit{reductio ad absurdum} focused on an inconvertible indexical dilemma ("the relation \( \theta \)-corresponding \textit{to the same couple}'). And the real target of so irreverent a Great Theorem is just to emphasize the essential presupposition which every sound \textit{reductio} is grounded on, that is the non-defectivity of the respective dilemma. In other words, (18.xxxviii) is a misleading notation since \( GN \) is an indexical relation, and as soon as we indicate it by "\( GN^s \)" we realize that an auto-conversion is a pseudo-promotion to absoluteness, because it leads to an expression where the reflexive variable continues occurring.
A18.1. Contemporary set theories struggle in well known difficulties. Quine (1963, Preface) writes: the axiomatic systems of set theory ... are largely incompatible ... no one of them clearly deserves to be singled out as standard ... intuition here is bankrupt. A rather disconcerting conclusion, indeed, not only because the notion of a set seems highly intuitive. but also because a crucial question imposes: what ought the choice of axioms be dictated by, if not by our intuition?

A18.2. The proposed solution of Russell’s paradox (§18.6.2), bans many absurd theorems of the current set theories (first among them the mentioned theorem according to which \( \{x: x=x\} \) is empty). Another source of paradoxes is represented by arguments depending on notions like *all*, *the greatest* et cetera. The following suggestion might be useful.

A18.3. A set \( z \) is well defined over a universe of reference \( \Omega \) iff for every individual \( x \) of \( \Omega \) the definition allows us to decide whether \( x \) is a member of \( z \).

A set can be defined either intensionally (by establishing the characteristic an individual must possess in order to be a member of the set under formation) or extensionally (by listing its members). Since extensional definition are quite unproblematic, henceforth only intensional definitions are considered.

A18.4. In compliance with Cantor’s original position, a set is a collection of entities of any sort (Suppes 1972, footnote p.1); therefore a collection of sets, too. The peculiarity characterizing the sets of sets is that while a set of apples is not an apple, a set of sets is a set. A basic distinction opposes the sets of sets that while a set of apples is not an apple, a set of sets is a set. A basic distinction opposes the sets of sets such that

\[(A18.i) \quad P\{x: Px\}\]

(let me call them “open”) to the sets of sets such that

\[(A18.ii) \quad \neg P\{x: Px\}\]

(let me call them “closed”). Referring such a distinction to sets of sets is a necessary condition to assure the properness of (A18.i), where the same predicate is ascribed to \( x \) and to \( \{x: Px\} \).

The opposition between open and closed sets can be directly extrapolated to the respective conditions of membership (that is to their intensions).

The successor \( m' \) of an open set \( m \) is the set whose members are the same \( m \) and the members of \( m \).

A18.5. Obviously the set of those individuals that are \( P \), besides depending on \( P \), depends on the universe of reference. Shoenfield (in Barwise 1977, §2) writes: When we are forming a set \( z \) ... we do not yet have the object \( z \). I agree totally: no set can pre(exist to its birth. This means that, given a basic universe of reference \( \Omega' \), when we are dealing with an open set \( m \), the universe resulting from the union of \( \Omega \) and \( m \) is a new universe \( \Omega' \) different from \( \Omega \) (and so on). In this sense

\[(A18.iii) \quad \{x: Px\}\]

is a correct notation only if the set is closed, that is iff (18.ii) holds. Otherwise (18.iii) is an ambiguous notation where it is not specified if we are referring to \( \Omega' \) or to its successor \( \Omega' \) (and so on); in this sense open sets entail diachronically mutable universes of reference. And quantifying over a diachronically mutable universe without specifying the moment our quantification refers to is introducing a heavy lack of information; for instance an expression like

No \( x \) exists such that …..

opens spontaneously the way to the question: when does not it exist? Realizing that what does not exist at \( t' \) (that is with reference to \( \Omega' \)) may exist at \( t' \) (that is with reference to \( \Omega \)) is the essential acquirement in order to avoid an incumbent risk of preposterousness. And it is easy to verify that all impasses concern open sets; for instance the membership requisite for the set of all sets is simply to be a set.

A18.5.1. For the sake of concision and in conformity with previous assumptions I only consider increasing universes. So if something does not exist at \( t \), then it cannot exist in any preceding moment, and if something does exist at \( t \), then it does exist in any following moment.

A18.6. As far as I know a meticulous respect of our intuition leads to a sound and unproblematic diachronic set theory, that is to a theory whose basic rule is avoiding any preposterousness by exacting the chronologic specification of every open set. Therefore my suggestion is simple: diachronizing formally the various expressions through qualified indexes. So for instance
names the set of those $x$ that at $t''$ are $P$,

$\exists x(Px)$

says that at $t'$ no $x$ exists such that $Px$; analogously

$\exists' x(Px)$

abbreviates

$\exists x(Px) \& \neg \exists y(y \neq x \& Py)$

and so on. Under this convention, for instance,

$\exists' x(Px) \& \neg \exists \tau x(Px)$

can be interpreted as the birth at $t+1$ of exactly one $x$ such that $Px$.

A18.6-1. An immediate example of the advantages offered by such a diachronization is supplied by

$(A18.\text{iv})$

representing the axiomatic scheme of abstraction in its new formulation. And actually $(A18.\text{iv})$ precludes any paradox: for instance,

$\{x: x \notin x\}' \in \{x: x \notin x\}'$

far from being self-contradictory, represents exactly our intuitive conviction.

A18.6.2. Let me insist. A notation like $(A18.\text{iii})$ is admissible for closed sets because, written concisely,

$\neg P\{x: Px\}^o \supset (\{x: Px\}^o = \{x: Px\}')$

that is because the formation of $(\{x: Px\}^o$ does not increase the set of sets satisfying the same condition of membership. On the other hand $(A18.\text{iii})$ is a notation to reject if the set is open because

$P\{x: Px\}^o \supset (\{x: Px\}^o \neq \{x: Px\}')$

and the lacking chronologic index then makes $(A18.\text{iii})$ a referentially ambiguous expression.

A18.7. I suppose that the well known opposition between sets and (proper) classes might be reduced to the opposition between open and closed sets. Indeed the current notion of classes is not well defined. On this subject Suppes (1972. §2-6) acknowledges that classes appear rather bizarre from the standpoint of naive, intuitive set theory. (cf. also Lake, 1974 p.415). And anyhow some basic incompatibilities among the intuitions of the most celebrated theorists of the distinction between classes and sets are undeniable. For instance the question may a class be identified with the set having the same members? would be answered negatively by Bernays and affirmatively by Goedel (1958, footnote 5). Of course somebody could object that intuition is an optional in an axiomatic theory; but I could not only evoke Kleene (An intuitive mathematics is necessary even to define the formal mathematics ... the ultimate appeal... must be to the meaning and evidence rather than to any set of conventional rules): I could also re-propose a heavy reply based on the admissibility criterion: what ought the axioms be legitimated by, if not by intuition?

In my opinion there is no need to assume the notion of classes among primitives. Nelson Goodman (1943, p. 107) writes

A given idea $A$ need to be left as primitive in a system only so long as we have discovered between $A$ and the other primitives no relationship intimate enough to permit defining $A$ in terms of them and I cannot but agree; yet in my opinion classes are open sets. And an open set can be the member of another set under the previous fixation of the moment of reference since, otherwise, the lack of a chronologic index would involve us in a referential ambiguity (in a lack of information). Could we correctly ask how many inhabitants had Rome without specifying the moment of reference? And are we compelled to specify the moment of reference when we ask how many edges has a cube?

Here too preposterousness is the root of the impasses.

A18.8. Preposterousness is an essential factor in diagonal procedures and Berry’s paradox is a symptomatic example. Shortly, such a paradox arises as soon as we realize that

$(A18.\text{v})$

the smallest number which cannot be succinctly defined is succinctly defined by $(A18.\text{v})$. Though its solution can be led back to the argument proposed in Chapter 18, let me face it through a peculiar analysis.

A number can be defined on the grounds of some specific connotation. Formally we must only appeal to connotations derived from primitive notions; for instance 3 is the successor of the successor of the successor of 0. Yet informally we can appeal to any connotation resulting from the statute of reference; for instance 3 is the number of Graces or of triumviri et cetera. Thus we list a set $m$ of definitions which, in its turn, can be assumed as evidence for new connotations. But if we appeal to a connotation like that, we are necessarily involved in an open context, and we then must avoid any confusion between the previous set $m$ and, speaking trivially, its successor $m'$ obtained by adding the definition we are performing to the same $m$. As such the smallest number which, with reference to $m$ cannot be succinctly defined, may be succinctly defined with reference to $m'$. In other and more
general words. Any quantification must be referred to the domain of the variable under quantification; and to speak of the least number which in \( m \) is not defined in a certain way is to say that actually in \( m \) there is no definition such that et cetera, therefore it is a quantification; yet of course a correct quantification over \( m \), may be incorrect over \( m' \). In this sense the argument recalls Richard’s original solution (I remind the reader Richard’s *at the place it occupies, it has no meaning*); in this sense Berry’s paradox too is born by a preposterous pseudologism.

A18.9. The informational approach, precisely because of the central role played by the knower, then by meanings (intensions, connotations, properties), rejects the exasperated extensionalism of the current theorizations of logic. Identifying an \( n \)-place relation with the respective set of \( n \)-uples is violating our most deep-rooted cognitive mechanisms. When I say that the matrimonial relation is dangerous I am not at all saying that the set of conjugal couples is dangerous; I am speaking of a link, not of a collection. Since two sets having the same members are the same set and since the set of individuals with a hearth is the set of individuals with a liver, how could a strictly extensional approach recognize their strong distinctive factor? The intensional step (inquiring into a characteristic) is very often necessary to ascertain that some individual is a member of some set. Therefore no valid theoretical approach can ignore that our mind avails itself of intensions not less than of extensions. I am not fighting against extensionalism, I am fighting against its pretension to be the only approach, a pretension which seems to me a desperate attempt to ennoble through a strict formalism a truly stone age metaphysical perspective.

A18.10. A last consideration entailing a metaphysical compromise focuses on the credentials on whose grounds the same relation \( \in \) of membership is assumed as a primitive notion. Reality is unitary. Only for gnosiogetic convenience we partition it in a multiplicity of individuals (and the necessary net of relations is the price to pay for restoring the unitarity). Of course, usually, our partitions are firmly suggested (but not logically imposed) by objective physical discontinuities (we see a rider and his horse as separate individuals, Aztecs saw *conquistadores* on horseback as single monsters, nobody sees the rider and half horse as a single individual). A set is nothing but an individual resulting from the assumption *ut unum* of a collection, that is of some otherwise autonomous individuals resulting from a previous partition; and such an assumption *ut unum*, roughly, is a sort of conjunction. In this sense \( \in \) can be reduced to a relation of identity between the member we are speaking of and one of the conjoined individuals (that is to a disjunction among identities between individuals). This viewpoint might also shed light on the hardly debated relation between individuals and singular sets.

Anyhow I emphasize that mine is a mere suggestion.
CHAPTER 19
INDEXICAL FUNCTIONS

19.1. Kleene (1974, p.24) writes: *by a system ... we mean a (non empty) set ... (or possibly several such sets) of objects among which are established certain relationships.* Since "system" seems to me a semantically overcharged word, I replace it by "structure" (symbolically "Ω"). Informally speaking, a structure is then a (usually schematic) universe whose individuals can be unequivocally identified through the net of relationship linking each of them to an individual assumed as primitively known.

Although the structure of main interest is \( N \), that is the set of natural numbers, I think that approaching the matter with reference to a less simple structure allows a wider and better understanding of the whole discourse. So Figure 19.1 represents partially this structure \( N \), a sort of dichotomic tree with one only number zero, two numbers one (the beta-one and the mu-one, so to write), four numbers two (the beta-beta two et cetera), and in general with \( 2^n \) numbers \( n \). Exactly as \( N \) is formally described by Peano’s axioms, \( N \) could be formally described by analogous axioms. Indeed such a formalization might be useful in some scientific fields, yet I prefer to follow a less irksome procedure; and to reason intuitively on the graph. In fact such a procedure is sufficient to reach the aim of this chapter: to show that the opposition *absolute* vs. *indexical* can be punctiliously extrapolated to functions.

19.2. In §15.13.2 I remarked (with reference to \( N \)) that under a severely formal approach, symbolizing numerical variables by something like "ξ0", "ψ0" et cetera (where "ξ" and "ψ" are variables standing for a concatenation of the primitive functor "f") would be better than symbolizing them by something like "x", "y" et cetera. Since the same remark holds for \( N \), it will be actually applied below.

19.3. First of all, some banal terminological agreement.

In order to emphasize the similarities between \( N \) and \( N \), I agree to call “number” every member of \( N \); too, “numeral” any term naming a number (as for instance "μββ0"), and “prefix” the part of a numeral preceding “0" (that is, in the case, the concatenation of primitive functors "μββ").

Precisely as the point βψ0 is the β-successor of the point ψ0 and the point ψ0 is the β-predecessor of the point βψ0, the point μψ0 is the μ-successor of the point ψ0 and the point ψ0 is the μ-predecessor of the point μψ0.

A pedantry: since every point (except 0) has only one predecessor, the specifications β-predecessor and μ-predecessor are superfluous.

A basic move is the passage from a point to one of its successors.

Two points are contiguous iff one of them is the successor of the other. A unitary interval is the distance between two contiguous points. A move is unitary iff it covers a unitary interval; therefore basic moves are unitary.

19.4. Given two points ξ0 and ψ0 on the graph, I call “route (between ξ0 and ψ0)” the shortest way leading from ξ0 to ψ0; obviously, once agreed that a way is loop-free iff no unitary interval is covered in both directions, any route is loop-free.

The crucial problem concerns the possibility of covering whatever route on the graph. And in order to achieve such a possibility we must be enabled to perform two fundamental operations on basic moves: the concatenation and the inversion. The concatenation enables us to cover an ordered sequence of basic moves, the
inversion enables us to cover back a basic move (concatenation and inversion are in \(N_2\) what addition and subtraction are in \(N_1\)).

For instance, the route \(\phi\) from \(a=\mu\beta\beta0\) to \(b=\mu\beta0\) is performed by covering back a \(\mu\)-interval (thus passing from \(\mu\beta\beta0\) to \(\beta\beta0\)), after by covering back a \(\beta\)-interval (thus passing from \(\beta\beta0\) to \(\beta0\)) and finally by covering forward a \(\mu\)-interval (thus passing from \(\beta0\) to \(\mu\beta0\)).

I call “connection” any relation like

\[ b = \Phi(a) \]

where “\(a\)” names the argument, “\(\Phi\)” names the route and “\(b\)” names the exargument (the reason why “exargument” is preferred to “value” will be explained in §19.8.3 below). For the sake of concision I only deal with monadic and monodrome connections.

19.5. An evident one-to-one correspondence does exist between routes and prefixes. As a matter of fact, any number is identified by its co(ordinate, and exactly as such a co(ordinate (if we reason on the graph) is the route to cover from 0 to the same number, such a co(ordinate (if we reason on expressions) is the sequence of primitive functors constituting the prefix of its numeral. In this sense diagrammatical and syntactical discourses are easily interchangeable; in particular I call “algorithm” the correspondent of a route, that is the prefix which, applied to the numeral for the argument, turns it into the numeral for the exargument.

19.5.1. While concatenation is formulated by writing in succession the concatenated functors, the inversion is formulated by raising to the unitary negative power the concatenation of the inverted functors. Obvious rules of simplification follow. For instance, once \(\alpha\) is assumed to represent the relation of identity, the rule

\[ (\xi)(\xi^{-1}) = \alpha \]

(concisely: \(\xi\xi^{-1} = \alpha\)) tells us that covering a route forward and back leads us to the same point we started from. Analogously

\[ (\xi\psi)(\psi^{-1})(\xi^{-1}) = \psi^{-1}\]

tells us that the first step of a covered back route is the last step of the same route et cetera.

19.6. Indeed (19.i) and (19.ii) are fragments of the systematic formal approach to \(N_2\) mentioned in §19.1. Yet, as \(N_2\) itself is nothing but an example, and an example whose peculiarities of specific interest can be emphasized without any appeal to a severe formal approach, here I limit myself to sketch very briefly such a formal approach.

First of all by

\[ \sim \exists (\beta 0 = 0) \]
\[ \sim \exists (\mu 0 = 0) \]

we say that 0 is the origin, by

\[ \alpha 0 = \xi 0 = \xi 0 \]

we introduce the relation of identity \(\alpha\). Furthermore by

\[ \sim \exists (\beta 0 - \mu 0) \]

we state that \(\beta\) and \(\mu\) are two distinct relations (stating \(\beta=\mu\) is reducing \(N_2\) to \(N_1\)).

By

\[ 0(\xi 0) = \xi(n-1)(\xi 0) \]

we define recursively the multiplication, by

**Theorem** \[ \alpha^{-1} 0 = \alpha 0 \]

**Proof** \[ \alpha \alpha^{-1} 0 = \alpha^{-1} \alpha 0 = \alpha 0 \]

we state that the inverse of the identity is again the identity. And so on.

19.7. While intensionalists conceive a function as a correspondence between arguments and exarguments (Kleene, ibidem, p.32: a function is a correspondence), extensionalists (Suppes, 1972, §3.1: this vague idea of intuitive connectedness may be dispensed) conceive a function as an ordered triple of sets respectively for the domain, codomain (image) and ordered couples. Generally reasoning, the extensional approach is lessened by some strong objections, and precisely:

a) the same set of ordered couples can be the extension of two or more different intensions
b) the same function can be intuitively extrapolated to totally different domains on the sole ground of its intensional characteristics (the mother function, say)
c) if a function were its extension, how could we have a precise idea of many functions (the mother function, say again) whose extension we know only for an infinitesimal fragment?

Indeed I believe that our way to elaborate information is intrinsically intensional because only the faculty of extrapolating intensions gives us the possibility to rule new situations by analogy. Under my informational
viewpoint, the extensional approach jumps to the results neglecting the factor ruling the formation of the pairs, that is the exact factor upon which the opposition between absolute and indexical functions is based (with a bit of malice I would evoke the Aesopian *nolo acerbam sumere*). Therefore, awaiting §19.11 where the extensional approach too will be considered, for the moment a function is a set of algorithms (of routes) and consequently a function is well defined (over a domain) iff the respective algorithm (the respective route) is established for any argument of the domain.

19.8. Let

\[(19.\text{iii})\]
\[\psi_0 = \Phi(\xi_0)\]
be a well defined function (over a certain domain). This means that for every specific \(\xi_0\) of the domain we know the respective algorithm \(\Phi\). For instance both the functions \(\Phi_1\) (assigning to every number its \(\mu\mu\)-successor) and \(\Phi_2\) (assigning to every number its double, that is the number whose prefix doubles the previous one) are well defined on \(N\), then both “\(\Phi_1\)” and “\(\Phi_2\)” name an unambiguous referent (so to say, every well defined function has the unquestionable right to possess a name). In other words, both

\[(19.\text{iv})\]
\[\psi_0 = \Phi_1(\xi_0)\]
and

\[(19.\text{v})\]
\[\psi_0 = \Phi_2(\xi_0)\]
are correct and unambiguous formulae. Nevertheless a crucial difference opposes (19.iv) to (19.v). In fact, as by definition (19.iv) is

\[(19.\text{vi})\]
\[\psi_0 = \beta\beta(\xi_0)\]
and (19.v) is

\[(19.\text{vii})\]
\[\psi_0 = \xi(\xi_0)\]
while “\(\Phi_1\)” stands for a free-variable-free concatenation of primitive symbols, “\(\Phi_2\)” stands for a free-variable-laden concatenation of primitive symbols. In fact, while “\(\Phi_1\)” stands always for “\(\beta\beta\)” quite independently on the argument it is applied to, “\(\Phi_2\)” stands always for “\(\beta\beta\)” when the argument is \(\beta\beta\), but it stands for “\(\mu\mu\)” when the argument is \(\mu\mu\). For instance both the functions “\(\Phi_2\)” and indexical functions (as \(\Phi_2\)). Of course such an opposition presupposes a formal approach to a universe structured by some relationships (that is a universe whose main example is \(N\)).

19.8.1. A better formulation of (19.vii) is

\[(19.\text{viii})\]
\[\psi_0 = s(\xi_0)\]
because the reflexive variable redeems the choice of the independent variable.

19.8.2. To call \(\Phi_2\) “doubling function” is an example of the point b) in §19.7, since it is the spontaneous extrapolation of what the doubling function \(2\bullet\) is in arithmetic. In fact (19.vii) can be identically applied to \(N\).

It is rather significant (minding again Presburger) to underline that, while additions are absolute functions, multiplications are indexical functions, and that a strong link can be detected between indexicality and recursivity, which actually draws from any contingent argument the piece of information necessary to single out the respective algorithm (I remind the reader that the formally adequate notation for variables allows a non-recursive definition of addition). That is: a simple generalization over (19.vii) and (19.vi) shows that while the functors for multiplication are free-variable-laden, the functors for addition are free-variable-free. For instance, while there is no concatenation of primitive functors “\(2\bullet\)” stands always for \(2\bullet\) (that is \((2\bullet)(\xi_0), 2\bullet(\psi_0)) = \psi(\psi_0)\)) and so on), each one of the four \(2+\) additions (that is \(\beta\beta, +\beta\mu, +\mu\beta, +\mu\mu\)) is an absolute function since \(\beta\beta(\xi_0) = \beta\beta\), exactly as \(\beta\beta(\psi_0) = \beta\beta\psi_0\) and so on. Course the four \(2+\) additions of \(N\) reduce themselves to the only and current \(2+\) addition of \(N\). Then “\(2+\)” and “\(2\bullet\)” are like notations for highly unlike operations.

19.8.3. Let (19.iii) be a well defined indexical function. The fixation of \(\xi_0\) (that is the assignation of a specified number \(\xi_0\) to the argument) entails the conversion of \(\Phi\) on \(\Phi^\sigma\) and consequently of \(\psi_0\) on \(\psi^\sigma\). Therefore the aim of avoiding any lexical confusion between the various \(\psi_0\) and the various \(\Phi\) induced me to identify the values of a function with the various \(\Phi\) (with the various algorithms) and to coin a specific term (“exargument”, I mean) for the various \(\psi_0\). In other words: to say that \(\Phi^\sigma\) is the value and \(\psi^\sigma\) is the exargument of the function (19.iii) for the argument \(\xi^\sigma\) is a quite spontaneous lexical choice in order to avoid any interpretative ambiguity.

19.9. It is well known that

\[(19.\text{viii})\]
\[\Phi(\xi)\]
that is, henceforth,
is an ambiguous expression. In my semantics (if ever I will succeed in publishing it) so serious an ambiguity is overcome by better articulated conventions; yet here I have to deal with the current situation, and in the current situation (19.viii) is used sometimes to speak of $\psi 0$ (exargumental reading) and sometimes to speak of $\Phi$ (algorithmic reading). Therefore if an identity like
\[
\Phi(\emptyset) = \psi(\emptyset)
\]
is algorithmically true ($\Phi = \psi$), it is tautological, and as such it is also exargumentally true; on the contrary an exargumentally true identity (Bob’s wife is Bob’s first love) does not imply any algorithmic identity. For instance
\[
\psi'(\psi(\emptyset)) = \alpha(\emptyset)
\]
is unquestionably true under its exargumental reading, since if we move from $\emptyset$ and, once reached $\psi\emptyset$ through an arbitrary $\psi$-route, we cover back the same route ($\psi'$), we find ourselves again in the point where we would find ourselves if we should have stayed there ($\alpha$). This notwithstanding (19.ix) is absolutely false under its algorithmic reading, since it is a trivially untenable claim to identify the set of programs
\[
(19.x)

\begin{align*}
& \text{go there and come back home} \\
& \text{(where “there” plays just the role of a variable played in (19.ix) by “$\psi$”) with the program}
\end{align*}
\]
\[
(19.xi)

\text{stay home}
\]
(where “stay” plays just the role of a constant played in (19.ix) by “$\alpha$”).

19.10. In an intriguing (yet neglected) paper whose knowledge is presupposed (English translation in Van Heijenoort 1967, p.355: *On the building blocks of mathematical logic*) Schoenfinkel reaches a puzzling formal result: nothing less than the total elimination of variables. In my opinion such a paper represents the insuperable top of an acritical formalism. Quine too, introducing it, refuses Schoenfinkel’s reduction (contrasting it with serious ones) yet he honestly admits that his refusal is not grounded on sound arguments; indeed he emphasizes as a risky passage to deal with functions whose arguments are functions (functionals), yet functionals are a current and manifestly is of no theoretical moment).

For instance (19.xi)
\[
\begin{align*}
& \text{a function}\[0.2cm]\text{(identity function) and}
\end{align*}
\]
\[
(19.xii)

Cxy = y
\]
(constancy functions, so calling any function whose exargument is always the same for every argument of the domain). Actually (19.xii) tells us that the constancy functions are a set, one for any fixed exargument; and precisely in order to point out this substantial difference between the two variables (Schoenfinkel would say that one of them is a blind one), let me (provisionally) assume
\[
(19.xiii)

\begin{align*}
& \text{to mean generically the constancy function (of $x$, obviously) whose exargument is $y$. So for instance, under such a (provisional)
\end{align*}
\]
\[
(19.xiv)

\begin{align*}
& \text{assumption}
\end{align*}
\]
is the particularization of (19.xiii) on the origin, that is the constancy function assigning the exargument 0 to any argument $x$. The translation of (19.xiv) into the alternative notation is
\[
(19.xv)

\begin{align*}
& \text{because applying the algorithm $\xi^{1}$ to whatever $\emptyset$ leads us to 0. Incidentally, (19.xiii), (19.xiv) and (19.xv) hold as well in $N_{i}$ as in $N_{2}$ (all depends on the set of individuals (numbers) the independent variable ranges over).}
\end{align*}
\]
However the momentous conclusion we draw is that, in spite of their name, constancy functions are indexical. Informally such a conclusion is dictated by the evidence that if we must anyway arrive at a previously fixed point quite independently on the point we move from, the route we must cover varies with the same starting point; formally the same conclusion is dictated by the evidence that “$\xi^{1}$” is a free-variable-laden algorithm. Here too the use of the reflexive variable would be better, since it would free us from the choice of the independent variable (in the context above, where “$\xi$” is the variable chosen to name the generic argument, “$\xi^{1}$” is already the conversion of “$s^{1}$”).

19.10.2. Therefore correcting (19.xiii) in something like
\[
(19.xvi)

\begin{align*}
& \text{is nothing but recognizing symbolically that “$C$” occurs in (19.xv) as an algorithmic variable, that is as a symbol}
\end{align*}
\]
\[
\text{standing for a sequence of primitive symbols which depends on the argument, too. The necessary presence of an “s” in (19.xvi) can be also argued as follows. If we start from (19.xiii), we get}
\]
\[
Iy = Cx
\]
\[
(19.xvii)

\begin{align*}
& \text{is the particularization of (19.xvi) on the origin, that is the algorithmic function assigning the exargument 0}
\end{align*}
\]
\[
(19.xviii)

\begin{align*}
& \text{to any argument $x$. The translation of (19.xvii) into the alternative notation is}
\end{align*}
\]
\[
(19.xix)

\begin{align*}
& \text{because applying the algorithm $\psi^{0}$ to whatever $\emptyset$ leads us to 0. Incidentally, (19.xvi), (19.xvii) and (19.xviii) hold as well in $N_{i}$ as in $N_{2}$ (all depends on the set of individuals (numbers) the independent variable ranges over).}
\end{align*}
\]
However the momentous conclusion we draw is that, in spite of their name, constancy functions are indexical. Informally such a conclusion is dictated by the evidence that if we must anyway arrive at a previously fixed point quite independently on the point we move from, the route we must cover varies with the same starting point; formally the same conclusion is dictated by the evidence that “$\psi^{0}$” is a free-variable-laden algorithm. Here too the use of the reflexive variable would be better, since it would free us from the choice of the independent variable (in the context above, where “$\psi$” is the variable chosen to name the generic argument, “$\psi^{0}$” is already the conversion of ““$s^{0}$”).
that is a formula affected by an unbalanced presence of “x” only in its right side. On the contrary if we start from (19.xvi), we get

\[ Iy = \xi C_x x \]

that is a formula where the mentioned presence of “x” is counterbalanced by the presence of “s”

In other words: since

\[ s^x(x(y)) = y \]

is an intuitively understandable alternative to (19.xii), to claim that in (19.xii) “C” is a constant contradicts the same definition of constancy functions, then confutes Schoenfinkel’s thesis.

The link between logical paradoxes and his reduction is that here too the presence of a free variable is misrecognized. And really eating variables is a quite trustworthy way to achieve their total elimination; persevering is enough.

19.10.3. Yet, even if we put apart such a very consequential formal abuse, we stumble over another one. Here it is. Undoubtedly (19.xvii) is a formally correct identity in its exargumental reading. In fact (let me insist through a banal instance inspired by (19.x) and (19.xi)) they who, being in the cathedral, stay there, and they who, wherever they are, come back home and go to the cathedral, find themselves in the same final place. Nevertheless (19.xvii) is absolutely false under its algorithmic reading since

\[ \text{stay in the cathedral} \]

and

\[ \text{wherever you are, come back home and go to the cathedral} \]

are evidently different programs. Symbolically, once assumed a given prefix \( \psi^\alpha \), might we reasonably claim that

\[ \alpha \psi^\alpha \]

and

\[ \psi^\alpha \psi^\xi \]

are the same algorithm?

But this algorithmic reading of formulae whose validity, at the most, would be strictly limited to their exargumental reading is a current practice in Schoenfinkel’s reduction (a blatant example in his §4, where \( I= SCC \) is inferred from \( Ix= SCCx \)).

19.11. His approach to functions is explicitly intensional (by function we mean a correspondence, he writes in §2). Anyhow, far from offering some reasonable way out, the extensional approach to functions converges perfectly with the above conclusions. For instance, with reference to

\[ f0 = \xi C_x x \]

it would be senseless to identify the pair \( \langle 0, 0 \rangle \) with the set of pairs \( \{ \langle 0, 0 \rangle, \langle 1, 0 \rangle, \langle 2, 0 \rangle \ldots \} \), though the second member of the pair is the second member of every pair belonging to the set. Analogously, with reference to (19.xvi), it would be senseless to identify the set of pairs \( \{ \langle 0, 0 \rangle, \langle 1, 1 \rangle, \langle 2, 2 \rangle \ldots \} \) with the set of set of pairs \( \{ \{ \langle 0, 0 \rangle, \langle 1, 0 \rangle, \langle 2, 0 \rangle \ldots \}, \{ \langle 0, 1 \rangle, \langle 1, 1 \rangle, \langle 2, 1 \rangle \ldots \}, \{ \langle 0, 2 \rangle, \langle 1, 2 \rangle, \langle 2, 2 \rangle \ldots \}, \ldots \} \), though et cetera.

19.11.1. I do not dwell on other absurdities derivable from Schoenfinkel’s work. The true usefulness of so unilateral a reduction is just showing how dangerous an excess of acritical formalism can be, and such a task seems to me already accomplished. Anyhow I do not intend to evoke the Rylean Back to ordinary language! I limit myself to evoke the classical Primum vivere, deinde philosophari, and to translate so wise a precept into something like: understanding before formalizing. Indeed here I hope nobody will ask me what I mean by “understanding”, because, democratically, my only answer would be: agreeing with my opinions.
20.1. It is well known that Gödel’s Über formal unentscheidbare Sätze der Principia Mathematica und verwandter Systeme (henceforth [G]) raised hard criticisms. Yet, as far as I know, all of them were affected by an intrinsic weakness: proposing general considerations without tackling the formal procedure. In my opinion this is the reason why Gödel himself neglected such criticisms as lucubrations of no theoretical interest. And really (Wittgenstein) contesting from a philosophical viewpoint the intelligibility of a formula strictly inferred through a constructive procedure or (Perelman) reasoning on the propaedeutically argumentative explanation of such a procedure are intrinsically weak approaches. I agree with their intuitions, I too think that Gödel’s undecidable formula $17Gen$ is not a proper formula, yet I avow that his First Theorem can be unquestionably confuted only by the exhibition of some formal mistake. This chapter achieves such a result.

20.1.1. Since I maintain *Italic type* in compliance with Gödel’s convention ([G] p. 179: *Diejenigen Klassen.....* in *Kursivschrift*), for the sake of clearness, the quotations are printed in Franklin Gothic. In order to help collations, sometimes reference is also made to van Heijenoort’s English translation (henceforth [H]). Anyhow, as far as reasonable, Gödel’s original terminology is adopted; so while “Satzformel” (“sentential formula” in [H]) means a free-variable-free formula “Klassenzeichen” (“class sign” in [H]) means a formula where exactly one free variable occurs, “Zahlzeichen” (“numeral” in [H]) means an expression formed by a certain number (also null) of “$f$” concatenated to a final “0”. Finally, for the sake of typographical simplicity, some inconsequential modifications are brought to Gödel’s notation for the substitution operator.

20.2. A crucial role is played by the notion of proper formulae ([G] p.174 *sinnvolle Formeln*; [H] p.597 *meaningful formulas*). The basic difference between proper and improper formulae is clear: while the oppositive negation of an untrue but proper (a false) formula must be true, the oppositive negation of an improper (then an untrue) formula is again an improper formula. For instance, since

- Odd(4) is untrue but proper
- Even(4) is true; on the other hand, since
- Odd($\pi$) is untrue and improper, also
- Even($\pi$) is untrue and improper.

20.2.1. My central claim can be informally epitomized as follows: a formula stating its own (un)provableness is improper because it violates the dialinguistic orders.

Indeed my claim seems to be immediately contradicted by two pieces of evidence, and precisely

- the formal procedure through which Gödel achieves his undecidable formula is constructive
- many other formal procedures leading to a formula stating its own (un)provability have been proposed.

Nevertheless a more severe analysis lets us understand that both pieces of evidence are misleading.

20.2.2. At first sight, even the distinction between proper and improper formulae may appear of no moment as for the matter under scrutiny; in fact, since natural numbers are the exclusive members of the arithmetized universe Gödel speaks of, all the notions he defines concern strictly natural numbers. In order to prove that such a conclusion too is misleading, I start reasoning before any arithmetization, that is with reference to what I call “standard ambience” (symbolically, “SA”). Of course I follow Gödel’s formal system $P$ which results from the insertion of Peano’s axioms within the logic of Principia Mathematica ([G] p.176; $P$ ist im wesentlichen das System...).

20.3. Let $L$ be the language for $P$ and $ML$ its metalanguage. Since defined symbols ([G] p.174, footnote *) are only abbreviations, a $P$-expression is a concatenation of primitive symbols belonging to $L$; I call “Peanic” such expressions. Analogously I call “syntactic” the expressions formed by a concatenation of $ML$-symbols, that is the expressions which, in their standard interpretation, speak of Peanic expressions.

I emphasize that all the following new lines introducing symbolic expressions have a metalinguistic import.

20.3.1. Although Gödel introduces the substitution operator in a wider context ([G] p.177: *Unter Subst* ...), I mainly focus on substitutions where the initial formula is a Klassenzeichen, and the free variable to substitute is replaced by a constant so that the final formula is a Satzformel.
20.3.2. The properness of a Klassenzeichen is implicitly presupposed by the domain of its free variable. Variables are signs, therefore necessary and sufficient condition for the properness of the Satzformel which we obtain by a substitution is unquestionable: the substituendum must name a value of the substituted variable, that is it must name a member of the domain the same variable ranges over. So, once assumed

\[(20.\text{i}) \quad x \quad y \quad z\]

as Peanian variables whose values are natural numbers, the names of the natural numbers, that is the Zahlzeichen (the numerals) are their proper substitutors. Analogously, once assumed

\[(20.\text{ii}) \quad g \quad v \quad n\]

as syntactic variables ranging over Peanian expressions (sentences, terms and so on), their proper substitutors are the metalinguistic expressions which name Peanian expressions of the right syntactical status.

20.4. The generic substitution operated on a Peanic Klassenzeichen is described by a $ML$-expression like

\[(20.\text{iii}) \quad \text{Subst}(g; \: v/n)\]

where

- the initial formula $g$ (without inverted commas, I am not speaking of the syntactic variable in boldface, I am using it to speak of the generic object formula the same syntactic variable stands for, as for instance “Prim($y$)”), is precisely a Peanic Klassenzeichen
- the substituendum $v$ (without inverted commas ...) is the free variable occurring in $g$ (“$y$”, in the previous instance), that is a variable ranging over natural numbers
- the substitutor $n$ (...) is the numeral naming a value of the free variable, that is a number (for instance “ffff0”).

Then

\[(20.\text{iv}) \quad \text{Subst}("Prim($y$)"; \: "y"/"ffff0")\]

is a proper $ML$-description of

\[(20.v) \quad \text{Prim}(ffff0)\]

and (20.v) is a (proper and false) Peanic Satzformel stating that 4 is a prime number.

Precisely because (20.v) is a proper formula,

\[(20.vi) \quad \text{Prim}("ffff0")\]

is an improper one. Reciprocally, once “Num” is assumed to symbolize the syntactic predicate “to be a numeral”,

\[(20.vii) \quad \text{Num}("ffff0")\]

is a proper formula and

\[(20.viii) \quad \text{Num}(ffff0)\]

is an improper one. And precisely because the improperness of (20.vi) and (20.vii) follows from a violation of the dialinguistic orders, I say that both formuale are affected by a projective improperness.

20.5. Analogously to (20.iii), the generic substitution operated on a syntactic Klassenzeichen speaking of object formuale is described by a $MML$-expression like

\[(20.viii) \quad \text{Subst}(G; \: V/E)\]

where

- the initial formula $G$ (that is, for instance, “Prov$_P$(y)”, once “Prov$_P$” is assumed to symbolize the predicate of provableness in $P$), is a syntactic Klassenzeichen
- the substituendum $V$ (“$y$” in the previous instance) is its syntactic free variable whose domain is formed by $P-$formulæ
- the substitutor $E$ (for instance “Prim(ffff0)” is the name of an “$y$”-value, that is the name of a $P$-formula.

Then, analogously to (20.iv),

\[(20.ix) \quad \text{Subst}("\text{Prov}_P(y)"; \: "y"/"Prim(ffff0)"")\]

is a proper $MML$-description of

\[(20.x) \quad \text{Prov}_P("\text{Prim}(ffff0)")\]

and (20.x), though false, is a proper syntactic Satzformel stating that the object Peanic Satzformel (20.v) is provable in $P$; of course (20.x) is false because actually the opposite of (20.v), that is

\(\sim(\text{Prim}(ffff0))\)

is provable in $P$, that is because the same (20.v) is refutable in $P$.

20.5.1. The absolute necessity to respect the dialinguistic orders is punctually satisfied by the above formuale. So, since (ix) is a $MML$-expression, both $ML$-expressions occur within a single pair of quotation marks and the only $L$-expression occurs within a double pair of quotation marks.

Let me insist. For instance (20.x) is a proper particularization of the proper syntactic Klassenzeichen

\[(20.xi) \quad \text{Prov}_P(y)\]

since (20.x), although false, is a proper metalinguistic formula stating that a well specified object Peanic formula (“Prim(ffff0)”, I mean) is provable in $P$. On the contrary

\[(20.xii) \quad \text{Prov}_P(\text{Prim}(ffff0))\]

is a projectively improper particularization of (20.xi), since in (20.xii) the object formula is used, not mentioned.
On the opposite side, also

\( \text{Subst}(\text{Prim}(00f)^y) \) (Def.11) is a Gödelian notion since it does depend on the arithmetization (the product of 2^{11}, 3^{17} and 5^{13} is the arithmetic image of "(x)" only because 11 has been co-ordinated (assigned) to "(" et cetera.

In other words: Peano’s axioms are sufficient to ascertain whether a certain number is a cube, but surely they are not sufficient to ascertain whether a number is a formula because in order to ascertain whether a number is a formula the resort to some arithmetization becomes a necessary step.

In other words. Although both of them concern numbers, the elusive but essential difference between Peanian and Gödelian notions is a difference of dialinguistic order because any arithmetization gives the numbers a double role: referents and signs.

20.8. Now I prove that

\( \text{Sb}(p; 19/Z(p)) \) is the proper description of an improper formula.

First of all I prove that, while being a numeral is a sufficient condition to being the proper substitutor of a variable ("variable" not in italic) whose values are numbers, being a numeral is not a sufficient condition to be the proper substitutor of a variable (though its values, obviously, are numbers). In fact, since

- a precise and computable Gödelian is co-ordinated to any (even improper) L-expression (for instance, the product of 2^1, 3^3 and 5^5 is the Gödelian for "00f"); therefore the distinction between proper and improper Gödelians is unquestionable;
- every Gödelian n (Def.17) has a numeral \( Z(n) \) (for instance the product of the first 750 prime numbers raised to the cube and of the 751st prime number is the numeral for the Gödelian image of "00f");

being a numeral is not a sufficient condition to be the proper substitutor of a variable.

Yet, as Gödel’s proof does not involve improper Gödelians (that is, with reference to §20.6: it does not involve any impropperness of first level), once their existence is recognized, we can neglect them.

Now I prove that even being the numeral for a proper Gödelian (that is for the image of a proper L-expression) is not a sufficient condition to be the proper substitutor of a variable. In fact, for instance, if the values of the variable to substitute are formulae, the numeral for a numeral is a manifestly improper substitutor, just as "ffff0" is an improper
substitutor of the free variable in (20.xi) (what does it mean to state that the number 4 is provable in \( P \)?) Yet, as Gödel’s proof does not involve this second level of improperness, we can neglect it, too.

Finally (and here is the crucial passage) I prove that even being the numeral for the Gödelian image of a formula is not a sufficient condition to be the proper substitutor of a variable whose values are formulae. In fact if the values of the variable are object formulae and the substitutor is the numeral for the Gödelian image of a syntactic formula, the general and fundamental condition is violated according to which, in order to obtain a proper Satzformel from a proper Klassenzeichen, the substitutor of the free variable must name a value of the same variable. This is just the case of (20.xv), since by definition ([G] p.188, formula (9)) \( p \) is a Klassenzeichen whose range is constituted by object formulae while \( Z(p) \), consequently, is not the numeral for an object formula.

Indeed, though (20.xv) is a number we can actually compute, its computability is far from entailing its properness. Exactly as (20.xiv) is the proper description of the improper (20.xiii), (20.xv) is the proper description of an improper Satzformel.

With reference to §20.6.1, we see that computable Gödelians correspond to every level of improperness.

20.8.1. The immediate intuitive understanding of this argument is restrained by the fact that usually we reason about numbers in SA, where actually being a numeral is a sufficient condition to be the proper substitutor of a numerical variable. But as soon as numbers are assumed as signs, we charge them with further (syntactical, so written) duties. Therefore, the properness of a Gödelian obtained by substitution can be assured only by the fact that the substitutor is the numeral for a Gödelian of the right sortal range, that is for a value of the substituted variable.

20.9. Gödel ([G], p.189: ...큰 effektiv aufweisbare) Satzformel 17Gen r....; [H] p. 609 ...the sentential formula...) claims that

\[
(20.xvi) \quad \text{17Gen} \, r
\]

is a (proper) Satzformel; therefore, ([G] p.188, formula (13), [H] p. 608) since (20.xvi) and (20.xv) are equivalent, he claims that (20.xv) is a (proper) Satzformel. But in order to prove his claim it is not at all sufficient to remark (ibidem) that \( p \) is a Klassenzeichen with the free variable 19) since his remark, at the most, could only succeed in assuring that no free variable occurs in (20.xv), where the free variable has been substituted by a numeral. First of all Gödel should prove that \( Z(p) \) is a proper substitutor of 19, then he should prove that \( Z(p) \) is the numeral of a formula (which is true) and, above all, he should prove that \( Z(p) \) is the numeral of a formula belonging to the domain of 19 (which is untrue). Substituting in the Klassenzeichen \( p \) the free variable 19, whose range is constituted by object formulae, with the numeral of a formula (as \( p \) actually is) which refers to the unprovability of an object formula means falling into the same projective mistake affecting (20.xiii): therefore the final formula cannot be a proper Satzformel.

As far as I understand, Gödel is not even touched by any suspicion about the properness of his undecidable formula precisely because, missing the distinction between Peanic and syntactic formulae, he tacitly presupposes that substituting a variable ranging over formulae with a numeral for a formula cannot but transform the initial sinvolle Klassenzeichen into a sinvolle Satzformel. And just this presupposition is the unwitting trick by which eighty years of close investigations have been misled, particularly because such a presupposition was seemingly legitimated by the formal mistake focused in §11 below.

20.9.1. Let me resume. To claim at the same time that
- 19 is the free variable, occurring in the class sign \( p \);
- 19 can be properly substituted by the numeral for \( p \)
is to fall into a (projective) contradiction because \( p \) cannot be a value of its own free variable.

20.9.2. The proof that, if (20.xvi) were provable, then
\[
(20.xvii) \quad \text{Neg}(17\text{Gen} \, r)
\]
would be provable, too, does not represent a surprising result ([G] p.176: überraschenden Resultaten; [H] p. 599) but the obvious consequence of the projective improperness affecting (20.xv), and therefore (20.xvi). Ex absurdo quodlibet.

Aphoristically: what Gödel actually proves is not the incompleteness of the system, but the improperness of his undecidable formula.

20.9.2.1. In order to help the intuitive understanding, the situation can be visualized through two concentric circles where opposite formulae are represented by a pair of points corresponding in a polar symmetry. So if we agree that the circular crown A represents the improper formulae, the interior circle B (split diametrically in \( B_1 \) and \( B_2 \) respectively for provable and refutable formulae) represents the proper ones. It would be an astonishing result to prove that if the point representing 17Gen \( r \) should fall into \( B_2 \), then its symmetric could not fall into \( B_2 \). But as soon as we realize that, on the contrary, the point representing 17Gen \( r \) falls into \( A \), the fact that its symmetric cannot fall into \( B_2 \) is an obviousness, since it too falls into \( A \). So the puzzle vanishes.

20.10. The just ascertained improperness of 17Gen \( r \) bears immediately, so to say, an intriguing meta-puzzle: how can an improper formula intrude into a formal system whose axioms are proper and whose transformation rules are
properness-conservative? The detailed answer, focusing the formal mistake through which such an intrusion is accomplished, represents the best validation of the above analysis. Here it is.

20.10.1. First of all I remind the reader (§15.14) that the choice of variables must respect two fundamental rules:

**R1:** not to choose the same variable for non-necessarily-identical numbers

**R2:** not to choose different variables for necessarily identical numbers.

The formal mistake we are pursuing depends exactly on a violation of **R1** (and a violation of the worst kind, since the same variable is chosen for necessarily-non-identical numbers).

20.10.2. In SA (that is: before any arithmetization), the Theorem of Representability STR (in two words: every recursive relation is representable) can be formulated (with reference to a dyadic arithmetic relation \( R \)) by

\[
(20.\text{xviii})
\]

\[
\text{Rec}(R) \rightarrow \exists S_{x,w} ((R(x,y) \rightarrow \text{Prov}_P (\text{Subst}(S_{x,w}; v/\text{nu}(x) w/\text{nu}(y))) & \\
& \& \neg R(x,y) \rightarrow \text{Prov}_P (\text{Subst}(\neg S_{x,w}; v/\text{nu}(x) w/\text{nu}(y)))
\]

where

- “Rec” is the predicate of recursivity
- \( S_{x,w} \) is a binary sign of relation with the free variables \( v \) and \( w \) (without inverted commas, analogously to (iii) I am not speaking of the syntactic variables in boldface, I am using them to speak of two generic object variables as for instance “\( u \)” and “\( z \)”)
- \( \text{nu}(x) \) and \( \text{nu}(y) \) are the numerals for the numbers \( x \) and \( y \).

Since I peacefully admit both the recursivity of the relations involved in Gödel’s proof and the existence of the respective sign of relation, and since the extrapolation from \( R(x,y) \) to \( \neg R(x,y) \) is immediate, I simplify (20.\text{xviii}) in its apodosis

\[
(20.\text{xix})
\]

\[
R(x,y) \rightarrow \text{Prov}_P (\text{Subst}(S_{x,w}; v/\text{nu}(x) w/\text{nu}(y))
\]

remarking that while the protasis of (20.\text{xix}) is formulated in \( L \) (it speaks of numbers) the apodosis is formulated in \( ML \) (it speaks of \( L \)-expressions). Just to mean that protasis and apodosis belong to different dialinguistic orders I say that STR is a projective theorem.

20.10.3. As long as we are in SA, the symbols occurring in the protasis of (20.\text{xix}) belong to \( L \) and the symbols occurring in its apodosis belong to \( ML \); therefore any risk is banned of violating **R1** through some abusive identification between the variables occurring in the protasis and the variables occurring in the apodosis. Of course avoiding choices which might be sources of superficial misunderstandings would be a welcome agreement; so, since in (20.\text{xix}) \( x \) and \( y \) are already the generic numbers we are speaking of in the protasis and whose numerals we are speaking of in the apodosis, choosing just “\( x \)” and “\( y \)” as values of \( v \) and \( w \) would be a rather spiteful decision. Nevertheless, strictly, such a decision too is formally unobjectionable:

\[
(20.\text{xx})
\]

\[
R(x,y) \rightarrow \text{Prov}_P (\text{Subst}(S_{x,y}; x/\text{nu}(x) y/\text{nu}(y))
\]

is a formally correct formulation because no abusive identification is possible between \( x \) and \( y \) (which are numbers) and “\( x \)” or “\( y \)” (which are symbols). In (20.\text{xx}) the only connection between protasis and apodosis continues consisting in that the number of “\( f \)” concatenated in the numeral substitutor of the free variable “\( x \)” is just \( x \) and that the number of “\( f \)” concatenated in the numeral substitutor of the free variable “\( y \)” is just \( y \).

Yet arithmetization changes radically the context.

20.11. Gödel’s Theorem V ([G] p.186: Satz V: Zu jeder rekursiven relation...; [H] p.607: Theorem V. For every recursive...) is the arithmetization (ATR) of STR. So, since in ATR both protases and apodoses speak of numbers, all the variables Theorem V deals range over numbers.

This simple consideration shows that, with all its apparent plausibility, what Goedel claims in his footnote 38 (ibidem) is abusive. The choice of variables is not at all arbitrary; in fact the risk does exist of violating **R1** through some formally illegitimate identification among the variables of the protasis and the variables of the apodosis. In his formulae (3) e (4), that is, shortly, in

\[
R(x, y) \rightarrow \text{Sb}(\text{Subst}(S_{x,y}; x/\text{nu}(x) y/\text{nu}(y))
\]

such a risk is implicitly avoided by the resort to different symbols (“\( x \)” and “\( u \)” and by the tacit presupposition that none of the “\( x \)”s occurring in the protasis identifies itself with some of the “\( u \)”s occurring in the apodosis. Yet such a risk is not at all avoided in the application of Theorem V to his formulae (9) and (10) ([G] p.188; [H] p.608) where the same 19 which in the protases occurs as the free variable of the formula \( \gamma \), in the apodoses occurs as a variable whose range is constituted by formulae like \( \gamma \). Therefore (9) and (10) are improper exactly as an SA formula where the same variable stands for an object formula in its first occurrence and for a syntactic formula in its second occurrence. Here is the formal mistake no authentic orthodoxy can accept; a (projective) mistake whose consequence is exactly the (projective) improperness of (20.xx). In fact should the illegitimate choice of the same free variable 19 not be used to carry out the proof, such a choice could be forgiven as a notational flaw of no theoretical moment. But this is not the case. In order to obtain (ibidem)

\[
\text{Sb}(p; 19/\text{Z}(p)) = ... = 17\text{Genr}
\]
Goedel applies (11) and (12) to the apodoses of (9) and (10), therefore he assumes that the range of the free variable 19 is constituted by formulae like $y$, but in order to obtain (15) and (16) he uses (13) for a substitution in the protases, thus he activates the projectively illegitimate identification the improprieness of his undecidable formula is born by.

20.11.1. Another (and very concise) way to realize the improprieness of (9) and (10) is to remark that $Z(y)$ cannot be at the same time the proper substitutor of the free variable 19 occurring in the protases and of the free variable 19 occurring in the apodoses, since the two ranges are separated by a dialinguistic order. Here is the rabbit Goedel pulls out of a hat, contrary to what Humphries (1979, p.539) thinks.

20.12. The only formally detailed proof I know is Goedel’s original one; yet I had the opportunity to read many other concise attempts at proving his First Theorem. Indeed to confute them is a quite superfluous task, since, in spite of any arithmetization, the documented projective improprieness of a formula stating its own (un)provability implies that some incorrect passage hides in every procedure leading to an analogous formula. And in effect an equivalent projective mistake can be found out in all of them. Let me analyse briefly two celebrated attempts.

20.12.1. The incorrect passage disqualifying Shoenfield’s argument concerns the proof of Church’s theorem (Shoenfield 1967 §6.8). The assumption of natural numbers as signs entails the already discussed consequence that not every numeral can be the proper substitutor of a variable. Then, since a necessary condition for the properness of $P(a,b)$ is that $a$ and $b$ belong to two consecutive dialinguistic orders, (more specifically: if $b$ is a syntactic formula, $a$ must be a Peanic one) the definition $Q(a) \leftrightarrow P(a,a)$ is improper: (~) TeorT is not recursive simply because it is improper.

20.12.2. Smullyan’s diagonalization (Smullyan 1993, Chapter 1) is the glorification of improprieness. He argues under the presupposition that diagonalization is an always legitimate operation: $H(h)$ is the diagonalization of $H$ and $H(h)$ is a sentence he explicitly claims (p.50). Probably the responsibility of his untenable presupposition depends also on the predicate (20.xxi) is read by John he repeatedly proposes as a privileged example. But (20.xxi) is a highly particular predicate (a dialinguistically polivalent predicate, so to say) since any expression, quite independently of any consideration about its dialinguistic order, can be read by John. In other words: since any text belongs to the sortal range of (20.xxi), its assumption as the subject of such a predicate yields a proper and properly diagonalizable sentence. But of course many predicates do not yield properly diagonalizable sentences; for instance while Prim ($x$) is proper, (20.xxii) $\text{Prim} (\text{"Prim} (x)\text{"})$ is improper. And to claim that such an improprieness is legitimated by some arithmetization means to contradict the same isomorphism because, in this case, we could exhibit some improper formula whose isomorphic image is a proper formula. Therefore, first of all, Smullyan ought to prove that (20.xi) is properly diagonalizable, which is not, since (20.xiii) is not less improper than (20.xxii).

20.12.2.1. A pedantry. Indeed a difference does exist between the improprieness of (20.xiii) and of (20.xxii); in fact, contrary to “Prim”, “Prov”, once mutilated of its reference to the system ($P$, in the case), is extrapolable to any dialinguistic order (provided the axiomatization of the corresponding system). Yet such a difference is theoretically negligible because, obviously, speaking of provability without specifying the axiomatic system of reference is an elliptic formulation totally inadmissible in a formal procedure.

20.13. Both the Liar and 17Gen $r$ hide the same incoherence: the identification of an object sentence with the metalinguistic sentence attributing a certain predicate (of falsity in the Liar, of refutability in the present case) to the same object sentence. Arithmetization is exactly the attempt to avoid this logically unavoidable hiatus.

20.14. Of course the fall of Gödel’s First Incompleteness Theorem entails the fall of the consequences he draws ([G] p.191: Wir zichen nun aus Satz VI weitere Folgerungen...). The general consideration (and the formal mistake affecting Schönfinkel’s reduction is another symptomatic example) is that formalism is a powerful weapon rather hard to deal with. Particularly because, until the logical mistake is not recognized, we tend to venerate an improper but formally inferred statement as a supremely profound achievement. A tendency not so strange as it may appear: in fact improprieness disconcerts our mind, and such a disconcertment may be interpreted as the extreme difficulty in understanding some transcendent truth.
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